Exercise 10a : Calculus with Parametric Curves

In Exercise (1)-(4), find an equation of the tangent line to the curve at the given point.

(1) \( x = t^4 + 1, \ y = t^3 + t; \ t = -1 \)
(2) \( x = e^{\sqrt{t}}, \ y = t - \ln t^2; \ t = 1 \)
(3) \( x = 2\sin 2t, \ y = 2\sin t; \quad (\sqrt{3}, 1) \)
(4) \( x = \sin t, \ y = \sin(t + \sin t); \quad (0, 0) \)

In Exercise (5)-(8), find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \). For which values of \( t \) is the curve concave upward?

(5) \( x = 4 + t^2, \ y = t^2 + t^3 \)
(6) \( x = t - e^t, \ y = t + e^{-t} \)
(7) \( x = 2\sin t, \ y = 3\cos t, \ 0 < t < 2\pi \)
(8) \( x = \cos 2t, \ y = \cos t, \ 0 < t < \pi \)

(9) Let \( P \) be a point at a distance \( d \) from the center of a circle of radius \( r \). The curve traced out by \( P \) as the circle rolls along a straight line is called a **trochoid**.

(a) Using the same parameter \( \theta \) as for the cycloid and assuming the line is the \( x \)-axis and \( \theta = 0 \) when \( P \) is at one of its lowest points, i.e. \((0, r - d)\), show that parametric equations of the trochoid are

\[
\begin{align*}
x &= r\theta - d\sin \theta \\
y &= r - d\cos \theta.
\end{align*}
\]

(b) Find the slope of the tangent line to the trochoid in terms of \( \theta \).

(c) Show that if \( d < r \), then the trochoid does not have a vertical tangent.

(10) At what points on the curve

\[
\begin{align*}
x &= t^3 + 4t \\
y &= 6t^2
\end{align*}
\]

is the tangent parallel to the line with equations \( x = -7t, \ y = 12t - 5 \)?
(11) Find equations of the tangents to the curve

\[ x = 3t^2 + 1 \quad y = 2t^3 + 1 \]

that pass through the point (4,3).

In Exercise (12)-(15), find the length of the given curve.

(12) \[ x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leq t \leq 1 \]

(13) \[ x = \frac{t}{1+t}, \quad y = \ln(1+t), \quad 0 \leq t \leq 2 \]

(14) \[ x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leq t \leq \pi \]

(15) \[ x = e^t - t, \quad y = 4e^{t/2}, \quad -8 \leq t \leq 3 \]

In Exercise (16)-(19), find the area of the surface obtained by rotating the given curve about the specified line.

(16) \[ x = t^3, \quad y = t^2, \quad 0 \leq t \leq 1, \quad \text{about the } x\text{-axis} \]

(17) \[ x = a \cos^3 \theta, \quad y = a \sin^3 \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad \text{about the } x\text{-axis} \]

(18) \[ x = 3t^2, \quad y = 2t^3, \quad 0 \leq t \leq 5, \quad \text{about the } y\text{-axis} \]

(19) \[ x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leq t \leq 1, \quad \text{about the } y\text{-axis} \]