Weighted semi-trapezoidal approximations of fuzzy numbers

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Abstract

Recently, many scholars investigated interval, triangular, and trapezoidal approximations of fuzzy numbers. These publications can be grouped into two classes: Euclidean distance class and non-Euclidean distance class. Most approximations in Euclidean distance class can be calculated by formulas, but calculating approximations in the other class is more complicated. In the present paper, we study a special class of non-linear approximations with respect to a weighted Euclidean distance. We call it “weighted semi-trapezoidal approximations”. The proposed approximations generalize all recent approximations in Euclidean distance class. First, we embed fuzzy numbers into a Hilbert space. Then compute weighted semi-trapezoidal approximations by means of best approximations in a closed convex subset of the Hilbert space. Finally, we propose formulas of matrix type, which is more clear than the previous contributions.

Keywords: Fuzzy numbers; Approximation; Hilbert space

1. Introduction

An arbitrary fuzzy number \( \tilde{A} \) can be represented by an ordered pair of left continuous functions \([A_L(t), A_U(t)]\), \( t \in [0,1] \), which satisfy the following conditions:

1. \( A_L \) is increasing on \([0,1]\),
2. \( A_U \) is decreasing on \([0,1]\),
3. \( A_L(t) \leq A_U(t) \), for all \( t \in [0,1] \).

The representation \( \tilde{A} = [A_L(t), A_U(t)], t \in [0,1], \) is called \( x \)-cuts of \( \tilde{A} \). Let \( R \) and \( F(R) \) denote the sets of real numbers and fuzzy numbers, respectively. \( \tilde{A} \in F(R) \) is said to be trapezoidal if its \( x \)-cuts are of the form

\[
[A_L(t), A_U(t)] = [a - \sigma(1-t), b + \beta(1-t)], \quad a \leq b \text{ and } \sigma, \beta \geq 0. \tag{1.1}
\]

Furthermore, a trapezoidal fuzzy number is said to be triangular if \( a = b \), and to be interval if \( \sigma = \beta = 0 \).

Let \( s_L, s_R > 0 \). In 2008, Nasibov and Peker [23] proposed a more general trapezoidal fuzzy number as follows:

\[
[A_L(t), A_U(t)] = [a - \sigma(1-t)^{1/s_L}, b + \beta(1-t)^{1/s_R}], \quad a \leq b \text{ and } \sigma, \beta \geq 0, \tag{1.2}
\]
which is called \((s_L, s_R)\) semi-trapezoidal in the present paper. Similarly, an \((s_L, s_R)\) semi-trapezoidal fuzzy number is \((s_L, s_R)\) semi-triangular if \(a = b\) holds, i.e.

\[
[A_L(t), A_U(t)] = [c - \sigma(1 - t)^{1/s_L}, c + \beta(1 - t)^{1/s_R}], \quad c \in \mathbb{R} \text{ and } \sigma, \beta \geq 0. \tag{1.3}
\]

If \(s_L = s_R = 1\) then (1.2) and (1.3) are traditionally trapezoidal and triangular fuzzy numbers, respectively.

Let \(d(\cdot, \cdot)\) be an arbitrary distance on \(F(\mathbb{R})\), and \(\tilde{A} \in F(\mathbb{R})\). The \((s_L, s_R)\) semi-trapezoidal (resp., \((s_L, s_R)\) semi-triangular, trapezoidal, triangular, symmetric triangular, interval) approximation of \(\tilde{A}\) with respect to the distance \(d(\cdot, \cdot)\) is the \((s_L, s_R)\) semi-trapezoidal (resp., \((s_L, s_R)\) semi-triangular, trapezoidal, triangular, symmetric triangular, interval) fuzzy number which minimizes the distance \(d(\tilde{A}, \tilde{X})\), where \(\tilde{X}\) is an \((s_L, s_R)\) semi-trapezoidal (resp., \((s_L, s_R)\) semi-triangular, trapezoidal, triangular, symmetric triangular, interval) fuzzy number. For linear shaped approximations of fuzzy numbers, Chanas [11] and Grzegorzewski [15] independently proposed their interval approximations. Afterwards, Ma et al. [22] proposed the symmetric triangular approximation, Delgado et al. [12] and Abbasbandy et al. [2-4] proposed their trapezoidal approximations with respect to different distances, Grzegorzewski and Mrówka [16,17] proposed trapezoidal approximations preserving the expected interval, and Abbasbandy and Hajjari [5] proposed trapezoidal approximations preserving cores of a fuzzy number. For non-linear shaped approximations of fuzzy numbers, in 2006 Hassine et al. [21] proposed a piece-wise parabolic approximation, in 2008 Nasibov and Peker [22] proposed the \((s_L, s_R)\) semi-trapezoidal approximation, and Ban [8] improved Nasibov and Peker’s approximation. In 2009, Grzegorzewski and Stefani [19] considered a quite general approximation. Unfortunately, many scholars had pointed that some approximations may fail to be fuzzy numbers [6,7,9,25]. For filling the gap, many authors proposed their improved approximations [7,8,17,18,20,26,27]. Many important properties and algorithms of trapezoidal approximations were proved by Grzegorzewski [18,20].

Let \(\lambda_L = \lambda_L(t)\) and \(\lambda_U = \lambda_U(t), t \in [0, 1]\), be nonnegative functions such that

\[
\int_0^1 \lambda_L(t) \, dt > 0 \quad \text{and} \quad \int_0^1 \lambda_U(t) \, dt > 0.
\]

In the present paper they are called weighted functions on \([0, 1]\). The weighted \(L_2\)-distance \(d_2(\cdot, \cdot)\) on \(F(\mathbb{R})\) is defined as

\[
d_2(\tilde{A}, \tilde{B}) := \left[\int_0^1 |A_L(t) - B_L(t)|^2 \lambda_L(t) \, dt + \int_0^1 |A_U(t) - B_U(t)|^2 \lambda_U(t) \, dt\right]^{1/2}. \tag{1.4}
\]

In [14], Grzegorzewski first proposed two families of distances on \(F(\mathbb{R})\): \(\delta_{p,q}\) and \(\rho_p\). In particular, if \(1 \leq p < \infty\), \(\delta_{p,q}\) is defined as follows:

\[
\delta_{p,q}(\tilde{A}, \tilde{B}) := \left[ (1 - q) \int_0^1 |A_L(t) - B_L(t)|^p \, dt + q \int_0^1 |A_U(t) - B_U(t)|^p \, dt \right]^{1/p}, \quad \text{where } 0 < q < 1.
\]

If we choose \(\lambda_L(t) = 1 - q\) and \(\lambda_U(t) = q\), we get

\[
d_2(\tilde{A}, \tilde{B}) = \delta_{2,q}(\tilde{A}, \tilde{B}).
\]

In 2007, Zeng and Li [30] proposed weighted triangular approximations of fuzzy numbers with respect to distance (1.4) for \(\lambda_L = \lambda_U = \lambda(t)\) such that \(\int_0^1 \lambda(t) \, dt = \frac{1}{2}\). Then Yeh [26,29] improved their approximations and proposed weighted trapezoidal approximations of fuzzy numbers. In the present paper, we consider weighted \((s_L, s_R)\) semi-trapezoidal and weighted \((s_L, s_R)\) semi-triangular approximations of fuzzy numbers (i.e. \((s_L, s_R)\) semi-trapezoidal and \((s_L, s_R)\) semi-triangular approximations with respect to weighted \(L_2\)-distance (1.4)). These approximations generalize both results in [8,29].

This paper is organized as follows. In Section 2 we introduce best approximations in Hilbert spaces (i.e. completely inner product spaces). In Section 3 we embed fuzzy numbers into a Hilbert space. In Section 4 we compute weighted \((s_L, s_R)\) semi-trapezoidal approximations of fuzzy numbers by applying the reduction principle [13, p. 80]. In Section 5 we compute weighted \((s_L, s_R)\) semi-triangular approximations of fuzzy numbers. In Section 6 we quote three examples to illustrate the importance of our results. In Section 7 we study properties of the proposed approximations.
2. Preliminaries: best approximations in Hilbert spaces

By an inner product space we mean a (real) vector space $V$ equipped with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying the following axioms:

1. $\langle u, u \rangle \geq 0$ for all $u \in V$, and $\langle u, u \rangle = 0$ iff $u = 0$,
2. $\langle u, v \rangle = \langle v, u \rangle$, for all $u, v \in V$,
3. $(au + bv, w) = a\langle u, w \rangle + b\langle v, w \rangle$, for all $u, v, w \in V$ and all $a, b \in \mathbb{R}$.

The inner product defines a metric
d

\[ d(u, v) := \langle u - v, u - v \rangle^{1/2}. \]

A completely inner product space is often called a Hilbert space. Let $\Omega$ be a subset of a Hilbert space $V$. We say that:

1. $\Omega$ is a subspace iff $u + v \in \Omega$ and $ru \in \Omega$ for all $u, v \in \Omega$ and all $r \in \mathbb{R}$,
2. $\Omega$ is convex iff $ru + (1 - r)v \in \Omega$ for all $u, v \in \Omega$ and all $r \in [0, 1]$,
3. $\Omega$ is Chebyshev iff for each $u \in V$ there exists a unique element $P_\Omega(u) \in \Omega$ such that

\[ d(u, P_\Omega(u)) \leq d(u, x), \quad \forall x \in \Omega, \]

and then $P_\Omega(u)$ is called the best approximation of $u$ in $\Omega$.

**Fact 2.1** (Deutsch [13, pp. 23–24]). Every closed convex subset (closed subspace, finite dimensional subspace) is Chebyshev.

**Fact 2.2** (Yeh [29, Fact 2.1]). Let $\Omega$ be a closed convex subset of a Hilbert space $V$ and $u \in V$. Then $P_\Omega(u)$ is the vector such that

\[ \langle u - P_\Omega(u), x - P_\Omega(u) \rangle \leq 0 \]

for all $x \in \Omega$. Moreover, $d(P_\Omega(u), P_\Omega(v)) \leq d(u, v)$ for all $u, v \in V$.

Note that Fact 2.2 implies that the best approximation $P_\Omega(\cdot)$ is always continuous. If $\Omega$ is a closed subspace then $P_\Omega(u)$ equals the projection of $u$ onto $\Omega$. Moreover

\[ \langle u - P_\Omega(u), x \rangle = 0, \quad \forall x \in \Omega. \]

**Fact 2.3** (Yeh [28, Appendix C]). Let $v_1, \ldots, v_m$ and $u$ be given vectors in an inner product space, and let $\sum_{i=1}^m t_i v_i$ be the projection of $u$ onto the subspace spanned by $\{v_1, \ldots, v_m\}$. Then

\[ (t_1, \ldots, t_m) \Phi = (\langle u, v_1 \rangle, \ldots, \langle u, v_m \rangle), \]

where $\Phi$ is the $m \times m$ matrix defined by $\Phi_{ij} = \langle v_i, v_j \rangle$.

If $v_1, \ldots, v_m$ are linearly independent, then $\Phi$ is invertible and positively definite, hence

\[ (t_1, \ldots, t_m) = (\langle u, v_1 \rangle, \ldots, \langle u, v_m \rangle) \Phi^{-1}. \]

By applying Cramer’s Rule we obtain

\[ t_i = \frac{1}{\det \Phi} \left| \begin{array}{cccc} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{i-1,1} & \Phi_{i-1,2} & \cdots & \Phi_{i-1,m} \\ \langle u, v_1 \rangle & \langle u, v_2 \rangle & \cdots & \langle u, v_m \rangle \\ \Phi_{i+1,1} & \Phi_{i+1,2} & \cdots & \Phi_{i+1,m} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{m1} & \Phi_{m2} & \cdots & \Phi_{mm} \end{array} \right|, \quad 1 \leq i \leq m. \]  

(2.1)
Fact 2.4 (The reduction principle [13, p. 80]). Let \( \Omega \) be a closed convex subset of an inner product space and \( V \) be any Chebyshev subspace that contains \( \Omega \). Then we have

\[
P_\Omega(u) = P_\Omega(P_V(u)).
\]

Fact 2.4 characterizes the best approximation \( P_\Omega(u) \) and provides a method for computing \( P_\Omega(u) \): First, find the subspace \( V_\Omega \) spanned by \( \Omega \), then compute the projection of \( u \) onto \( V_\Omega \). If the projection \( P_{V_\Omega}(u) \) belongs to \( \Omega \) we obtain

\[
P_\Omega(u) = P_\Omega(P_{V_\Omega}(u)) = P_{V_\Omega}(u).
\]

Otherwise, we shall consider the reduced problem: the best approximation of \( P_{V_\Omega}(u) \) in \( \Omega \).

3. Embedding fuzzy numbers into a Hilbert space

Let \( \lambda = \lambda(t) \) be a weighted function on \([0, 1]\). It is well-known that the set of all \( L^2 \)-integrable functions is a Hilbert space, denoted by \( L^2([0, 1]) \), on which the inner product is defined by

\[
(f, g) := \int_0^1 f(t)g(t)\lambda(t)\,dt.
\]

Another important Hilbert space is the product space. Let \( \lambda_L = \lambda_L(t) \) and \( \lambda_U = \lambda_U(t) \) be two weighted functions on \([0, 1]\). Define an inner product on the product space \( L^2_L \times L^2_U \) as

\[
((f_L, f_U), (g_L, g_U))_\lambda := \int_0^1 f_L(t)g_L(t)\lambda_L(t)\,dt + \int_0^1 f_U(t)g_U(t)\lambda_U(t)\,dt,
\]

for all \((f_L, f_U) \) and \((g_L, g_U) \) in \( L^2_L \times L^2_U \) on \([0, 1]\). Hence we obtain

\[
d_\lambda^2((f_L, f_U), (g_L, g_U)) = (f_L - g_L, f_U - g_U, (f_L - g_L, f_U - g_U))_\lambda
\]

\[= \int_0^1 |f_L(t) - g_L(t)|^2\lambda_L(t)\,dt + \int_0^1 |f_U(t) - g_U(t)|^2\lambda_U(t)\,dt.
\]

Recall that a fuzzy number \( \tilde{A} \) can be represented by an ordered pair of left continuous functions \([A_L, A_U]\) which satisfy the following conditions:

1. \( A_L \) is increasing on \([0, 1]\),
2. \( A_U \) is decreasing on \([0, 1]\),
3. \( A_L(t) \leq A_U(t), \) \( t \in [0, 1] \).

Because every monotonic function is integrable, \( A_L = A_L(t) \) and \( A_U = A_U(t) \) are both in \( L^2 \) for any weighted function \( \lambda = \lambda(t) \) on \([0, 1]\). In particular,

\[
(A_L, A_U) \in L^2 \times L^2 \times [0, 1] \times L^2_U \times [0, 1] \times L^2_U \times [0, 1].
\]

So we can regard \( \tilde{A} = [A_L(t), A_U(t)] \) as an element of \( L^2 \times L^2 \times [0, 1] \times L^2_U \times [0, 1] \) by the following identification

\[
i : \tilde{A} = [A_L(t), A_U(t)] \rightarrow (A_L, A_U) \in L^2 \times L^2 \times [0, 1] \times L^2_U \times [0, 1].
\]

In the present paper we always use the interval notation \([A_L, A_U]\) instead of \((A_L, A_U)\). In brief, we write it as follows:

\[
\tilde{A} = [A_L(t), A_U(t)] \in L^2 \times L^2 \times [0, 1] \times L^2_U \times [0, 1].
\]

Let \( \tilde{A} = [A_L(t), A_U(t)] \), \( \tilde{B} = [B_L(t), B_U(t)] \) in \( F(\mathbb{R}) \), and \( r \in \mathbb{R} \). The fuzzy addition \( \oplus \), fuzzy subtraction \( \ominus \), and fuzzy scalar multiplication \( \odot \) on \( F(\mathbb{R}) \) are defined as follows:

\[
\tilde{A} \oplus \tilde{B} := [A_L(t) + B_L(t), A_U(t) + B_U(t)],
\]

\[
\tilde{A} \ominus \tilde{B} := [A_L(t) - B_L(t), A_U(t) - B_U(t)],
\]

\[
\tilde{A} \odot r := [A_L(t) \cdot r, A_U(t) \cdot r].
\]
This shows that we can embed fuzzy numbers into the Hilbert space $L_r^2[0, 1]$, denoted by “$+$”, and its inverse operation (vector subtraction), denoted by “$-$”, is not equal to the fuzzy subtraction. In fact, the vector difference between two fuzzy numbers is:

$$\tilde{A} - \tilde{B} := [A_L(t) - B_L(t), A_U(t) - B_U(t)],$$

where $\tilde{A} = [-1 + 2t, 2 - t]$ and $\tilde{B} = [0, 2 - 2t] \in F(\mathbb{R})$, $t \in [0, 1]$, which are both triangular. Then, $\tilde{A} - \tilde{B} = [-1 + 2t, t] \notin F(\mathbb{R})$. But, it is easy to verify that, $\tilde{A} - \tilde{B} \in L_r^2[0, 1] \times L_u^2[0, 1]$, for all $\tilde{A}$ and $\tilde{B} \in F(\mathbb{R})$. The vector scalar multiplication on $L_r^2[0, 1] \times L_u^2[0, 1]$ is defined as follows:

$$r \cdot \tilde{A} := [rA_L(t), rA_U(t)],$$

where $\tilde{A} = [A_L(t), A_U(t)] \in L_r^2[0, 1] \times L_u^2[0, 1]$ and $r \in \mathbb{R}$. If $\tilde{A} \in F(\mathbb{R})$ and $r \geq 0$, it is easy to verify

$$r \cdot \tilde{A} = r \circ \tilde{A}.$$

On the contrary, for $r < 0$ we do not get the above equality.

From (1.4), we find that

$$d_\tilde{A}^2(\tilde{A}, \tilde{B}) = \langle \tilde{A} - \tilde{B}, \tilde{A} - \tilde{B} \rangle.$$

This shows that we can embed $F(\mathbb{R})$ into the Hilbert space $L_r^2[0, 1] \times L_u^2[0, 1]$ preserving fuzzy addition and fuzzy scalar multiplication by a nonnegative real number, i.e.

$$\tilde{A} \oplus \tilde{B} = \tilde{A} + \tilde{B} \quad \text{and} \quad r \circ \tilde{A} = r \tilde{A} \quad \text{if} \quad r \geq 0.$$

Hence we define an inner product on $F(\mathbb{R})$ inheriting from $L_r^2[0, 1] \times L_u^2[0, 1]$, that is

$$\langle \tilde{A}, \tilde{B} \rangle := \int_0^1 A_L(t)B_L(t)\lambda_L(t)dt + \int_0^1 A_U(t)B_U(t)\lambda_U(t)dt.$$

4. Weighted $(s_L, s_R)$ semi-trapezoidal approximations

Let $\mathcal{T}_{s_L, s_R}$ denote the set of $(s_L, s_R)$ semi-trapezoidal fuzzy numbers. In the previous section, we have embedded fuzzy numbers into the Hilbert space $L_r^2[0, 1] \times L_u^2[0, 1]$. From (1.2), we find that each element $\tilde{A} = [A_L(t), A_U(t)] \in \mathcal{T}_{s_L, s_R}$ can be expressed in the following form:

$$[A_L(t), A_U(t)] = a[1, 0] - \sigma[1 - t]^{1/s_L}, 0] + b[0, 1] + \beta[0, (1 - t)^{1/s_R}],$$

where “$+$” and “$-$” are vector operations, $a \leq b$, and $\sigma, \beta \geq 0$. Further on each element in $L_r^2[0, 1] \times L_u^2[0, 1]$ is called a vector. Now, let us fix four following vectors:

$$\tilde{E}_1 := [1, 0], \quad \tilde{E}_2 := [(1 - t)^{1/s_L}, 0], \quad \tilde{E}_3 := [0, 1], \quad \tilde{E}_4 := [0, (1 - t)^{1/s_R}].$$

Note that $\tilde{E}_i, 1 \leq i \leq 4$, are linearly independent and each element $\tilde{A} \in L_r^2[0, 1] \times L_u^2[0, 1]$ belongs to $\mathcal{T}_{s_L, s_R}$ iff

$$\tilde{A} = \sum_{i=1}^4 x_i \tilde{E}_i \quad \text{and} \quad x_2 \leq 0, \ x_4 \geq 0, \ x_1 \leq x_3.$$

Proposition 4.1. $\mathcal{T}_{s_L, s_R}$ is closed and convex.
Proof. Let \( \tilde{A} = \sum_{i=1}^{4} x_i \tilde{E}_i \) and \( \tilde{B} = \sum_{i=1}^{4} x'_i \tilde{E}_i \) be in \( \mathbb{T}_{s_L,s_R} \), and \( 0 \leq r \leq 1 \). Then,
\[
r\tilde{A} + (1 - r)\tilde{B} = \sum_{i=1}^{4} [rx_i + (1 - r)x'_i] \tilde{E}_i.
\]
By (4.2), \( x_2, x'_2 \leq 0, x_4, x'_4 \geq 0, x_1 \leq x_3 \) and \( x'_1 \leq x'_3 \) and therefore
\[
rx_2 + (1 - r)x'_2 \leq 0,
\]
\[
rx_4 + (1 - r)x'_4 \geq 0,
\]
\[
rx_1 + (1 - r)x'_1 \leq rx_3 + (1 - r)x'_3.
\]
Hence, \( r\tilde{A} + (1 - r)\tilde{B} \in \mathbb{T}_{s_L,s_R} \). This proves that \( \mathbb{T}_{s_L,s_R} \) is convex.

Let \( \tilde{A}_n = \sum_{i=1}^{4} x_i n \tilde{E}_i \in \mathbb{T}_{s_L,s_R}, n \in \mathbb{N} \), be a convergent sequence. So, it is a Cauchy sequence, i.e. for all \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that
\[
d_{\mathbb{L}}(\tilde{A}_n, \tilde{A}_m) = \langle \tilde{A}_m - \tilde{A}_n, \tilde{A}_m - \tilde{A}_n \rangle^{1/2} < \varepsilon
\]
for all \( m, n \geq N \). Define \( \Phi_{ij} = \langle \tilde{E}_i, \tilde{E}_j \rangle_{\mathbb{L}} \) and \( \Phi = [\Phi_{ij}] \). Since \( \tilde{E}_i, 1 \leq i \leq 4 \), are linearly independent, \( \Phi \) is symmetric and positively definite. Hence, each eigenvalue of \( \Phi \) is positive. Let \( r > 0 \) be the minimal eigenvalue. By applying bounds of Rayleigh quotient [1, pp. 181–182], we obtain
\[
r|x_{km} - x_{kn}|^2 \leq r \sum_{i=1}^{4} |x_{im} - x_{in}|^2 \leq \sum_{i,j=1}^{4} (x_{im} - x_{in})\Phi_{ij}(x_{jm} - x_{jn})
\]
for each \( 1 \leq k \leq 4 \). Note that
\[
\sum_{i,j=1}^{4} (x_{im} - x_{in})\Phi_{ij}(x_{jm} - x_{jn}) = \langle \tilde{A}_m - \tilde{A}_n, \tilde{A}_m - \tilde{A}_n \rangle_{\mathbb{L}} < \varepsilon^2.
\]
Hence, \( \{x_{kn}\} \) is a Cauchy sequence for each \( k \). Let
\[
\lim_{n \to \infty} x_{kn} = x_k, \quad 1 \leq k \leq 4.
\]
Then the limit of \( \tilde{A}_n \) is \( \tilde{A} = \sum_{i=1}^{4} x_i \tilde{E}_i \). By (4.2) we obtain
\[
x_2 = \lim_{n \to \infty} x_{2n} \leq 0, \quad x_4 = \lim_{n \to \infty} x_{4n} \geq 0, \quad x_1 = \lim_{n \to \infty} x_{1n} \leq \lim_{n \to \infty} x_{3n} = x_3.
\]
Hence, the limit \( \tilde{A} \) belongs to \( \mathbb{T}_{s_L,s_R} \). This proves that \( \mathbb{T}_{s_L,s_R} \) is closed. \( \square \)

According to Fact 2.1, Proposition 4.1 implies \( \mathbb{T}_{s_L,s_R} \) is Chebyshev, hence weighted \((s_L, s_R)\) semi-trapezoidal approximation of any fuzzy number \( \tilde{A} \in F(\mathbb{R}) \) exists and is unique. Let \( \mathcal{T}_{s_L,s_R}(\tilde{A}) \) denote the weighted \((s_L, s_R)\) semi-trapezoidal approximation of \( \tilde{A} \), \( \mathcal{V}_E \) denote the vector subspace spanned by \( \tilde{E}_i, 1 \leq i \leq 4 \), and \( \mathcal{V}_E(\tilde{A}) \) denote the projection of \( \tilde{A} \) onto \( \mathcal{V}_E \). Since \( \mathcal{V}_E \) is finitely dimensional, Fact 2.1 implies \( \mathcal{V}_E(\tilde{A}) \) exists and is unique. Since \( \mathbb{T}_{s_L,s_R} \subseteq \mathcal{V}_E \), by applying the reduction principle (Fact 2.4), we find
\[
\mathcal{T}_{s_L,s_R}(\tilde{A}) = \mathcal{T}_{s_L,s_R}(\mathcal{V}_E(\tilde{A})).
\]
(4.3)
Let \( \Phi = [\Phi_{ij}] \) be the \( 4 \times 4 \) matrix defined by
\[
\Phi_{ij} = \langle \tilde{E}_i, \tilde{E}_j \rangle_{\mathbb{L}}, \quad 1 \leq i, j \leq 4
\]
(4.4)
hence \( \Phi_{ij} = \Phi_{ji} \). From (4.1), we can easily verify
\[
\Phi_{ij} = \Phi_{ji} = 0, \quad i = 1, 2, \quad j = 3, 4.
\]
Since \((1 - t)^{1/L} \geq 0\) and \((1 - t)^{1/R} \geq 0\), \(t \in [0, 1]\), by (4.4) and (3.2) we get
\[
\phi_{ij} \geq 0, \quad \forall i, j.
\] (4.5)

The independence of \(\bar{E}_i\), \(1 \leq i \leq 4\), implies that \(\Phi = (\Phi_{11} \quad \Phi_{12} \quad 0 \quad 0)
\[
\begin{bmatrix}
\Phi_{11} & \Phi_{12} & 0 & 0 \\
\Phi_{21} & \Phi_{22} & 0 & 0 \\
\Phi_{33} & \Phi_{34} & 0 & 0 \\
\Phi_{43} & \Phi_{44} & 0 & 0
\end{bmatrix}
\]

are all invertible and positively definite. Hence
\[
\det \Phi > 0,
\] (4.6)
\[
\begin{align*}
\Phi_{11} & = \Phi_{11} - \Phi_{12}^2 > 0, \\
\Phi_{21} & = \Phi_{21} - \Phi_{22}^2 > 0, \\
\Phi_{33} & = \Phi_{33} - \Phi_{34}^2 > 0.
\end{align*}
\] (4.7)

By applying Fact 2.3, the projection of \(\tilde{A}\) onto \(\mathbb{V}_E\) is \(\mathcal{V}_E(\tilde{A}) = \sum_{i=1}^{4} x_i \bar{E}_i\), where \(x_i\) can be computed by
\[
(x_1, x_2, x_3, x_4) = (\langle \tilde{A}, \bar{E}_1 \rangle, \langle \tilde{A}, \bar{E}_2 \rangle, \langle \tilde{A}, \bar{E}_3 \rangle, \langle \tilde{A}, \bar{E}_4 \rangle) \Phi^{-1}.
\] (4.9)

**Lemma 4.2.** Let \(\tilde{A} \in F(\mathbb{R})\). Then,
\[
\langle \tilde{A}, \bar{E}_2 \rangle \Phi_{11} - \langle \tilde{A}, \bar{E}_1 \rangle \Phi_{12} \leq 0,
\] (4.10)
\[
\langle \tilde{A}, \bar{E}_4 \rangle \Phi_{33} - \langle \tilde{A}, \bar{E}_3 \rangle \Phi_{34} \geq 0.
\] (4.11)

**Proof.** Let \(\tilde{A} = [A_L(t), A_U(t)] \in F(\mathbb{R})\). The definition of fuzzy numbers implies \(A_L = A_L(t)\) is increasing and \(A_U = A_U(t)\) is decreasing. From (3.2), we find
\[
\langle \tilde{A}, \bar{E}_2 \rangle \Phi_{11} - \langle \tilde{A}, \bar{E}_1 \rangle \Phi_{12} = \int_0^1 A_L(t)(1 - t)^{1/L} \lambda_L(t) dt - \int_0^1 \lambda_L(t) dt \int_0^1 A_L(t) \lambda_L(t) dt - \int_0^1 (1 - t)^{1/L} \lambda_L(t) dt.
\]

The Chebyshev’s inequality [10] gives us
\[
\int_0^1 f(t) g(t) \lambda(t) dt \cdot \int_0^1 \lambda(t) dt \geq \int_0^1 f(t) \lambda(t) dt \cdot \int_0^1 g(t) \lambda(t) dt,
\]
for any increasing functions \(f = f(t)\) and \(g = g(t)\) and any weighted function \(\lambda = \lambda(t)\) (a simple proof for Chebyshev’s inequality is given in Appendix). By setting \(f(t) = A_L(t)\), \(g(t) = -(1 - t)^{1/L}\), and \(\lambda(t) = \lambda_L(t)\), we prove (4.10). In the same way, by setting \(f(t) = -A_U(t)\), \(g(t) = -(1 - t)^{1/R}\), and \(\lambda(t) = \lambda_U(t)\), we also prove (4.11). □

**Lemma 4.3.** Let \(\mathcal{V}_E(\tilde{A}) = \sum_{i=1}^{4} x_i \bar{E}_i\) be the projection of \(\tilde{A}\) onto \(\mathbb{V}_E\). Then, \(x_2 \leq 0\) and \(x_4 \geq 0\).

**Proof.** By (4.1) and (4.4) we find that
\[
\Phi = \begin{bmatrix}
\Phi_{11} & \Phi_{12} & 0 & 0 \\
\Phi_{21} & \Phi_{22} & 0 & 0 \\
0 & 0 & \Phi_{33} & \Phi_{34} \\
0 & 0 & \Phi_{43} & \Phi_{44}
\end{bmatrix}
\]

By (4.9) and Cramer’s Rule (see (2.1)) we obtain
\[
x_2 = \frac{1}{\det \Phi} \begin{vmatrix}
\Phi_{11} & \Phi_{12} & 0 & 0 \\
\Phi_{21} & \Phi_{22} & 0 & 0 \\
0 & 0 & \Phi_{33} & \Phi_{34} \\
0 & 0 & \Phi_{43} & \Phi_{44}
\end{vmatrix} = \frac{\Phi_{33} \Phi_{44} - (\langle \tilde{A}, \bar{E}_2 \rangle \Phi_{11} - \langle \tilde{A}, \bar{E}_1 \rangle \Phi_{12})}{\det \Phi}.
\]

By (4.6), (4.8) and (4.10) we prove \(x_2 \leq 0\). In the same way, by (4.6), (4.7) and (4.11) we also prove \(x_4 \geq 0\). □
**Theorem 4.4.** Let $\tilde{A} \in F(\mathbb{R})$ and $\mathcal{V}_E(\tilde{A}) = \sum_{i=1}^{4} x_i \tilde{E}_i$ be the projection of $\tilde{A}$ onto $\mathbb{V}_E$. If $x_1 \leq x_3$, then the weighted $(s_L, s_R)$ semi-trapezoidal approximation of $\tilde{A}$ is $\mathcal{V}_E(\tilde{A})$, otherwise it is equal to weighted $(s_L, s_R)$ semi-triangular approximation of $\tilde{A}$.

**Proof.** From Lemma 4.3 we find $x_2 \leq 0$ and $x_4 \geq 0$. If $x_1 \leq x_3$, by (4.2) we get $\mathcal{V}_E(\tilde{A}) \in \mathcal{T}_{s_L, s_R}$, hence $T_{s_L, s_R}(\mathcal{V}_E(\tilde{A})) = \mathcal{V}_E(\tilde{A})$. Consequently, by (4.3) we obtain

$$T_{s_L, s_R}(\tilde{A}) = T_{s_L, s_R}(\mathcal{V}_E(\tilde{A})) = \mathcal{V}_E(\tilde{A}).$$

Now, we assume $x_1 > x_3$. From (4.2) we find $\mathcal{V}_E(\tilde{A}) \notin F(\mathbb{R})$, hence

$$T_{s_L, s_R}(\tilde{A}) \neq \mathcal{V}_E(\tilde{A}).$$

By Fact 2.4, it needs to consider the best approximation of $\mathcal{V}_E(\tilde{A})$ in $\mathcal{T}_{s_L, s_R}$. Suppose on the contrary that the weighted $(s_L, s_R)$ semi-trapezoidal approximation

$$T_{s_L, s_R}(\tilde{A}) = \sum_{i=1}^{4} y_i \tilde{E}_i, \quad y_2 \leq 0 \quad \text{and} \quad y_4 \geq 0$$

is not $(s_L, s_R)$ semi-triangular, i.e. $y_1 < y_3$. Let $r = (y_3 - y_1)/(x_1 - x_3 + y_3 - y_1)$, then $r \in (0, 1]$. Consider the element

$$r\mathcal{V}_E(\tilde{A}) + (1 - r)T_{s_L, s_R}(\tilde{A}) = \sum_{i=1}^{4} [rx_i + (1 - r)y_i] \tilde{E}_i.$$

Since $x_2 \leq 0$ (by Lemma 4.3) and $y_2 \leq 0$ we get $rx_2 + (1 - r)y_2 \leq 0$. Similarly, $x_4 \geq 0$ and $y_4 \geq 0$ together imply $rx_4 + (1 - r)y_4 \geq 0$. Note that

$$rx_1 + (1 - r)y_1 = rx_3 + (1 - r)y_3.$$  

This shows that $r\mathcal{V}_E(\tilde{A}) + (1 - r)T_{s_L, s_R}(\tilde{A}) \in \mathcal{T}_{s_L, s_R}$. Since $T_{s_L, s_R}(\tilde{A})$ is the best approximation of $\mathcal{V}_E(\tilde{A})$ in $\mathcal{T}_{s_L, s_R}$, by applying Fact 2.2 we obtain

$$(\mathcal{V}_E(\tilde{A}) - T_{s_L, s_R}(\tilde{A}), r\mathcal{V}_E(\tilde{A}) + (1 - r)T_{s_L, s_R}(\tilde{A}) - T_{s_L, s_R}(\tilde{A}))_\lambda \leq 0.$$  

On the other hand, since $\mathcal{V}_E(\tilde{A}) \neq T_{s_L, s_R}(\tilde{A})$ we get

$$(\mathcal{V}_E(\tilde{A}) - T_{s_L, s_R}(\tilde{A}), r\mathcal{V}_E(\tilde{A}) + (1 - r)T_{s_L, s_R}(\tilde{A}) - T_{s_L, s_R}(\tilde{A}))_\lambda
= (\mathcal{V}_E(\tilde{A}) - T_{s_L, s_R}(\tilde{A}), r[\mathcal{V}_E(\tilde{A}) - T_{s_L, s_R}(\tilde{A})])_\lambda = r(\mathcal{V}_E(\tilde{A}) - T_{s_L, s_R}(\tilde{A}), \mathcal{V}_E(\tilde{A}) - T_{s_L, s_R}(\tilde{A}))- \lambda > 0,$$

which is a contradiction. Hence the weighted $(s_L, s_R)$ semi-trapezoidal approximation must be $(s_L, s_R)$ semi-triangular. $\square$

**5. Weighted $(s_L, s_R)$ semi-triangular approximations**

Let $\mathcal{T}_{s_L, s_R}^{A}$ denote the set of $(s_L, s_R)$ semi-triangular fuzzy numbers. Note that the element $\tilde{A} = \sum_{i=1}^{A} x_i \tilde{E}_i$ is in $\mathcal{T}_{s_L, s_R}^{A}$ iff $x_2 \leq 0$, $x_4 \geq 0$, and $x_1 = x_3$. Hence an element $\tilde{A} \in L^L_2[0, 1] \times L^U_2[0, 1]$ is in $\mathcal{T}_{s_L, s_R}^{A}$ iff

$$\tilde{A} = c(\tilde{E}_1 + \tilde{E}_3) + x \tilde{E}_2 + y \tilde{E}_4 \quad \text{and} \quad x \leq 0, \quad y \geq 0.$$  

By the same reasoning we obtain the following proposition.
Proposition 5.1. \( \mathbb{T}_{s_L, s_R}^A \) is closed and convex.

By Fact 2.1 and Proposition 5.1 \( \mathbb{T}_{s_L, s_R}^A \) is Chebyshev, hence weighted \((s_L, s_R)\) semi-triangular approximation of any fuzzy number \( \tilde{A} \in F(\mathbb{R}) \) always exists and is unique. We denote by \( A_{s_L, s_R}(\tilde{A}) \) the weighted \((s_L, s_R)\) semi-triangular approximation of \( \tilde{A} \). For computing the weighted \((s_L, s_R)\) semi-triangular approximation \( A_{s_L, s_R}(\tilde{A}) \), let us define three following subsets of \( F(\mathbb{R}) \):

\[
\Gamma_1 := \left\{ \tilde{A} \in F(\mathbb{R}) \left| \begin{array}{c}
\Phi_{12} \\
\langle \tilde{A}, \tilde{E}_2 \rangle \leq \langle \tilde{A}, \tilde{E}_4 \rangle
\end{array} \right| \right\}
\]

\[
\Lambda_1 := \left\{ \tilde{A} \in F(\mathbb{R}) \left| \begin{array}{c}
\Phi_{11} + \Phi_{33} \\
\langle \tilde{A}, \tilde{E}_1 + \tilde{E}_3 \rangle > -\frac{\Phi_{44}}{\Phi_{34}} \langle \tilde{A}, \tilde{E}_1 + \tilde{E}_3 \rangle \\
\langle \tilde{A}, \tilde{E}_2 \rangle \leq \langle \tilde{A}, \tilde{E}_4 \rangle
\end{array} \right| \right\},
\]

\[
\Lambda_2 := \left\{ \tilde{A} \in F(\mathbb{R}) \left| \begin{array}{c}
\Phi_{12} + \Phi_{34} \\
\langle \tilde{A}, \tilde{E}_2 \rangle \leq \langle \tilde{A}, \tilde{E}_4 \rangle
\end{array} \right| \right\},
\]

where \( \Phi_{ij}, 1 \leq i, j \leq 4, \) are defined by (4.4).

Lemma 5.2. Let \( \tilde{A} \in F(\mathbb{R}). \) Then

\[
\left| \frac{\Phi_{11} + \Phi_{33}}{\langle \tilde{A}, \tilde{E}_1 + \tilde{E}_3 \rangle} - \frac{\Phi_{12}}{\langle \tilde{A}, \tilde{E}_2 \rangle} \right| \leq 0
\]

and

\[
\left| \frac{\Phi_{11} + \Phi_{33}}{\langle \tilde{A}, \tilde{E}_1 + \tilde{E}_3 \rangle} - \frac{\Phi_{34}}{\langle \tilde{A}, \tilde{E}_4 \rangle} \right| \geq 0.
\]

Proof. Let \( \tilde{A} = [A_L(t), A_U(t)] \in F(\mathbb{R}). \) The definition of fuzzy numbers implies

\[
A_L(t) \leq A_L(1) \leq A_U(1) \leq A_U(t), \quad \forall t \in [0, 1].
\]

By (3.2) we get

\[
\frac{\langle \tilde{A}, \tilde{E}_2 \rangle}{\Phi_{12}} = \frac{1}{\int_0^1 A_L(t)(1-t)^{1/s_L} \lambda_L(t) dt} \leq A_L(1)
\]

and

\[
\frac{\langle \tilde{A}, \tilde{E}_3 \rangle}{\Phi_{33}} = \frac{1}{\int_0^1 A_U(t) \lambda_U(t) dt} \geq A_U(1).
\]

Since \( A_L(1) \leq A_U(1) \) we obtain

\[
\left| \frac{\Phi_{33}}{\langle \tilde{A}, \tilde{E}_3 \rangle} - \frac{\Phi_{12}}{\langle \tilde{A}, \tilde{E}_2 \rangle} \right| \leq 0. \text{ By Lemma 4.2 we find}
\]

\[
\left| \frac{\Phi_{11}}{\langle \tilde{A}, \tilde{E}_1 \rangle} + \frac{\Phi_{12}}{\langle \tilde{A}, \tilde{E}_2 \rangle} \right| \leq 0, \text{ so that}
\]

\[
\left| \frac{\Phi_{11} + \Phi_{33}}{\langle \tilde{A}, \tilde{E}_1 + \tilde{E}_3 \rangle} - \frac{\Phi_{12}}{\langle \tilde{A}, \tilde{E}_2 \rangle} \right| \leq 0.
\]

This proves (5.5). In the same way, we prove (5.6). \( \Box \)
Lemma 5.3. The three subsets, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, form a partition of $F(\mathbb{R})$.

Proof. By (5.2)–(5.4) we have $\Gamma_1 \cap \Gamma_2 = \Gamma_1 \cap \Gamma_3 = \emptyset$. Hence it suffices to prove $\Gamma_2 \cap \Gamma_3 = \emptyset$. Suppose on the contrary that there exists a fuzzy number $\tilde{A} \in \Gamma_2 \cap \Gamma_3$, that is

$$
\begin{vmatrix}
\phi_{12} & \phi_{34} \\
\langle \tilde{A}, \tilde{E}_2 \rangle_\lambda & \langle \tilde{A}, \tilde{E}_4 \rangle_\lambda
\end{vmatrix} > \begin{vmatrix}
\phi_{44} & \phi_{11} + \phi_{33} & \phi_{12} \\
\langle \tilde{A}, \tilde{E}_1 + \tilde{E}_3 \rangle_\lambda & \langle \tilde{A}, \tilde{E}_2 \rangle_\lambda & \langle \tilde{A}, \tilde{E}_3 \rangle_\lambda
\end{vmatrix}
$$

and

$$
\begin{vmatrix}
\phi_{12} & \phi_{34} \\
\langle \tilde{A}, \tilde{E}_2 \rangle_\lambda & \langle \tilde{A}, \tilde{E}_4 \rangle_\lambda
\end{vmatrix} > \begin{vmatrix}
\phi_{22} & \phi_{11} + \phi_{33} & \phi_{34} \\
\langle \tilde{A}, \tilde{E}_1 + \tilde{E}_3 \rangle_\lambda & \langle \tilde{A}, \tilde{E}_2 \rangle_\lambda & \langle \tilde{A}, \tilde{E}_4 \rangle_\lambda
\end{vmatrix}.
$$

By Lemma 5.2 we get

$$
\begin{vmatrix}
\phi_{12} & \phi_{34} \\
\langle \tilde{A}, \tilde{E}_2 \rangle_\lambda & \langle \tilde{A}, \tilde{E}_4 \rangle_\lambda
\end{vmatrix} > \begin{vmatrix}
\phi_{22} & \phi_{11} + \phi_{33} & \phi_{34} \\
\langle \tilde{A}, \tilde{E}_1 + \tilde{E}_3 \rangle_\lambda & \langle \tilde{A}, \tilde{E}_2 \rangle_\lambda & \langle \tilde{A}, \tilde{E}_4 \rangle_\lambda
\end{vmatrix} \geq 0.
$$

On the other hand, by computing the sum of (5.7) multiplied by $\phi_{12}^2 / \phi_{44}$ and (5.8) multiplied by $\phi_{12}^2 / \phi_{22}$ we obtain

$$
\left(\frac{\phi_{34}^2 + \phi_{12}^2}{\phi_{44}} + \frac{\phi_{12}^2}{\phi_{22}} - \phi_{11} - \phi_{33}\right) \begin{vmatrix}
\phi_{12} & \phi_{34} \\
\langle \tilde{A}, \tilde{E}_2 \rangle_\lambda & \langle \tilde{A}, \tilde{E}_4 \rangle_\lambda
\end{vmatrix} > 0.
$$

Hence

$$
\left(\frac{\phi_{34}^2 + \phi_{12}^2}{\phi_{44}} + \frac{\phi_{12}^2}{\phi_{22}} - \phi_{11} - \phi_{33}\right) > 0.
$$

From (4.7) and (4.8) we find $\phi_{12}^2 / \phi_{22} - \phi_{11} < 0$ and $\phi_{34}^2 / \phi_{44} - \phi_{33} < 0$. Then (5.10) contradicts (5.9). This proves $\Gamma_2 \cap \Gamma_3 = \emptyset$. □

Let $\mathcal{V}_A$ be the vector subspace spanned by $\tilde{E}_1 + \tilde{E}_3$, $\tilde{E}_2$, and $\tilde{E}_4$, and let $\mathcal{V}_A(\tilde{A})$ be the projection of $\tilde{A}$ onto $\mathcal{V}_A$. Since $\mathcal{V}_A$ is finitely dimensional, Fact 2.1 implies $\mathcal{V}_A(\tilde{A})$ always exists and is unique. Since $T_{sL,sR}^A \subseteq \mathcal{V}_A$ from Fact 2.4 we find

$$
A_{sL,sR}(\tilde{A}) = A_{sL,sR}(\mathcal{V}_A(\tilde{A})).
$$

Let

$$
\Psi := \begin{pmatrix}
\langle \tilde{E}_1 + \tilde{E}_3, \tilde{E}_1 + \tilde{E}_3 \rangle_\lambda & \langle \tilde{E}_1 + \tilde{E}_3, \tilde{E}_2 \rangle_\lambda & \langle \tilde{E}_1 + \tilde{E}_3, \tilde{E}_4 \rangle_\lambda \\
\langle \tilde{E}_2, \tilde{E}_1 + \tilde{E}_3 \rangle_\lambda & \langle \tilde{E}_2, \tilde{E}_2 \rangle_\lambda & \langle \tilde{E}_2, \tilde{E}_4 \rangle_\lambda \\
\langle \tilde{E}_4, \tilde{E}_1 + \tilde{E}_3 \rangle_\lambda & \langle \tilde{E}_4, \tilde{E}_2 \rangle_\lambda & \langle \tilde{E}_4, \tilde{E}_4 \rangle_\lambda
\end{pmatrix} = \begin{pmatrix}
\phi_{11} + \phi_{33} & \phi_{12} & \phi_{34} \\
\phi_{12} & \phi_{22} & 0 \\
\phi_{34} & 0 & \phi_{44}
\end{pmatrix}.
$$

Since $\tilde{E}_1 + \tilde{E}_3$, $\tilde{E}_2$, and $\tilde{E}_4$ are linearly independent, $\Psi$ is invertible and positively definite and

$$
\det \Psi > 0.
$$

According to Fact 2.3, for each $\tilde{A} = [A_L(t), A_U(t)] \in L_2^I \times L_2^I$, the projection of $\tilde{A}$ onto $\mathcal{V}_A$ is $\mathcal{V}_A(\tilde{A}) = c_\tilde{A}(\tilde{E}_1 + \tilde{E}_3) + x_\tilde{A}\tilde{E}_2 + y_\tilde{A}\tilde{E}_4$, where $c_\tilde{A}$, $x_\tilde{A}$, and $y_\tilde{A}$ are given by

$$
(c_\tilde{A}, x_\tilde{A}, y_\tilde{A}) = (((\tilde{A}, \tilde{E}_1 + \tilde{E}_3), (\tilde{A}, \tilde{E}_2), (\tilde{A}, \tilde{E}_4))\Psi^{-1}.
$$

(5.12)
Lemma 5.4. Let $\tilde{A} \in F(\mathbb{R})$ and $\mathcal{V}_A(\tilde{A}) = c_A(\tilde{E}_1 + \tilde{E}_3) + x_A\tilde{E}_2 + y_A\tilde{E}_4$ be the projection of $\tilde{A}$ onto $\mathcal{V}_A$. Then,

$$\Gamma_1 = \{ \tilde{A} \in F(\mathbb{R}) | x_{\tilde{A}} \leq 0 \text{ and } y_{\tilde{A}} \geq 0 \},$$

$$\Gamma_2 = \{ \tilde{A} \in F(\mathbb{R}) | x_{\tilde{A}} > 0 \},$$

$$\Gamma_3 = \{ \tilde{A} \in F(\mathbb{R}) | y_{\tilde{A}} < 0 \}.$$

Proof. By (5.12) and Cramer’s Rule we get

$$x_{\tilde{A}} = \frac{1}{\det \Psi} \begin{vmatrix} \Phi_{11} + \Phi_{33} & \Phi_{12} & \Phi_{34} \\ \Phi_{34} & 0 & \Phi_{44} \end{vmatrix} = \frac{\Phi_{34}}{\det \Psi} \begin{vmatrix} \Phi_{12} & \Phi_{34} \\ \Phi_{34} & \Phi_{44} \end{vmatrix} + \frac{\Phi_{44}}{\det \Psi} \begin{vmatrix} \Phi_{11} + \Phi_{33} & \Phi_{12} \\ \Phi_{34} & \Phi_{44} \end{vmatrix}$$

hence

$$\det \frac{\Psi}{\Phi_{34}}, x_{\tilde{A}} = \begin{vmatrix} \Phi_{12} & \Phi_{34} \\ \Phi_{34} & \Phi_{44} \end{vmatrix} + \frac{\Phi_{44}}{\det \Psi} \begin{vmatrix} \Phi_{11} + \Phi_{33} & \Phi_{12} \\ \Phi_{34} & \Phi_{44} \end{vmatrix}.$$ 

Since $\det \Psi > 0$ and $\Phi_{34} > 0$ we get

$$x_{\tilde{A}} > 0 \iff \begin{vmatrix} \Phi_{12} & \Phi_{34} \\ \Phi_{34} & \Phi_{44} \end{vmatrix} > -\frac{\Phi_{44}}{\det \Psi} \begin{vmatrix} \Phi_{11} + \Phi_{33} & \Phi_{12} \\ \Phi_{34} & \Phi_{44} \end{vmatrix}.$$ 

By (5.3) we prove $\Gamma_2 = \{ \tilde{A} \in F(\mathbb{R}) | x_{\tilde{A}} > 0 \}$. In the same way, we obtain

$$y_{\tilde{A}} < 0 \iff \begin{vmatrix} \Phi_{12} & \Phi_{34} \\ \Phi_{34} & \Phi_{44} \end{vmatrix} > \frac{\Phi_{44}}{\det \Psi} \begin{vmatrix} \Phi_{11} + \Phi_{33} & \Phi_{12} \\ \Phi_{34} & \Phi_{44} \end{vmatrix}.$$ 

By (5.4) we prove $\Gamma_3 = \{ \tilde{A} \in F(\mathbb{R}) | y_{\tilde{A}} < 0 \}$, too. Consequently, Lemma 5.3 implies $\Gamma_1 = F(\mathbb{R}) \setminus (\Gamma_2 \cup \Gamma_3)$. This completes the proof. 

Theorem 5.5. Let $\tilde{A} \in F(\mathbb{R})$. Then the weighted $(s_L, s_R)$ semi-triangular approximation $A_{s_L, s_R}(\tilde{A})$ of $\tilde{A}$ can be computed in the following cases:

1. If $\tilde{A} \in \Gamma_1$, then $A_{s_L, s_R}(\tilde{A}) = [c + x(1-t)^{1/s_L}, c + y(1-t)^{1/s_R}]$, where

$$c, x, y = ((\tilde{A}, \tilde{E}_1 + \tilde{E}_3), (\tilde{A}, \tilde{E}_2), (\tilde{A}, \tilde{E}_4))^{-1} \begin{pmatrix} \Phi_{11} + \Phi_{33} & \Phi_{12} & \Phi_{34} \\ \Phi_{12} & 0 & \Phi_{44} \\ \Phi_{34} & 0 & \Phi_{44} \end{pmatrix}.$$ 

2. If $\tilde{A} \in \Gamma_2$, then $A_{s_L, s_R}(\tilde{A}) = [c', c' + y'(1-t)^{1/s_R}]$, where

$$c', y' = ((\tilde{A}, \tilde{E}_1 + \tilde{E}_3), (\tilde{A}, \tilde{E}_4))^{-1} \begin{pmatrix} \Phi_{11} + \Phi_{33} & \Phi_{34} \\ \Phi_{34} & \Phi_{44} \end{pmatrix}.$$ 

3. If $\tilde{A} \in \Gamma_3$, then $A_{s_L, s_R}(\tilde{A}) = [c' + x'(1-t)^{1/s_L}, c']$, where

$$c', x' = ((\tilde{A}, \tilde{E}_1 + \tilde{E}_3), (\tilde{A}, \tilde{E}_2))^{-1} \begin{pmatrix} \Phi_{11} + \Phi_{33} & \Phi_{12} & \Phi_{34} \\ \Phi_{12} & 0 & \Phi_{44} \end{pmatrix}.$$
**Proof.** Let \( \mathcal{V}_A(\tilde{A}) = c(\tilde{E}_1 + \tilde{E}_3) + x\tilde{E}_2 + y\tilde{E}_4 \) be the projection of \( \tilde{A} \) onto \( \mathbb{V}_A \). By Lemma 5.4, \( \tilde{A} \in \Gamma_1 \) implies \( x \leq 0 \) and \( y \geq 0 \). By (5.1) we find \( \mathcal{V}_A(\tilde{A}) \notin F(\mathbb{R}) \). Hence, by (5.11) we obtain

\[
A_{sL, sR}(\tilde{A}) = A_{sL, sR}(\mathcal{V}_A(\tilde{A})) = \mathcal{V}_A(\tilde{A}).
\]

This proves Case (1) by applying (5.12).

To prove Case (2) we see that \( \tilde{A} \in \Gamma_2 \) implies \( x > 0 \) (by Lemma 5.4). Hence from (5.1) we find \( \mathcal{V}_A(\tilde{A}) \notin F(\mathbb{R}) \), that implies

\[
\mathcal{V}_A(\tilde{A}) \neq A_{sL, sR}(\tilde{A}).
\]

By Fact 2.4, it needs to consider the best approximation of \( \mathcal{V}_A(\tilde{A}) \) in \( \mathbb{T}_{sL, sR}^A \). From Lemma 5.3, \( \tilde{A} \in \Gamma_2 \) implies \( \tilde{A} \notin \Gamma_3 \), hence \( y \geq 0 \) (by Lemma 5.4). Let \( A_{sL, sR}(\tilde{A}) = c'(\tilde{E}_1 + \tilde{E}_3) + x'\tilde{E}_2 + y'\tilde{E}_4 \), where \( x' \leq 0 \) and \( y' \geq 0 \). Suppose on the contrary that \( y' < 0 \). Let \( r = (-x')/(x - x') \), then \( r \in (0, 1) \) and \( rx + (1 - r)x' = 0 \). Consider the element

\[
r\mathcal{V}_A(\tilde{A}) + (1 - r)A_{sL, sR}(\tilde{A}) = [rc + (1 - r)c'](\tilde{E}_1 + \tilde{E}_3) + [ry + (1 - r)y']\tilde{E}_4.
\]

Since \( y \geq 0 \) and \( y' \geq 0 \), we get \( ry + (1 - r)y' \geq 0 \). Eq. (5.1) implies

\[
r\mathcal{V}_A(\tilde{A}) + (1 - r)A_{sL, sR}(\tilde{A}) \in \mathbb{T}_{sL, sR}^A.
\]

Because \( A_{sL, sR}(\tilde{A}) \) is the best approximation of \( \mathcal{V}_A(\tilde{A}) \) in \( \mathbb{T}_{sL, sR}^A \), Fact 2.2 implies

\[
(\mathcal{V}_A(\tilde{A}) - A_{sL, sR}(\tilde{A}), r\mathcal{V}_A(\tilde{A}) + (1 - r)A_{sL, sR}(\tilde{A}) - A_{sL, sR}(\tilde{A}))_\lambda \leq 0.
\]

On the other hand, by \( \mathcal{V}_A(\tilde{A}) \neq A_{sL, sR}(\tilde{A}) \) we compute

\[
(\mathcal{V}_A(\tilde{A}) - A_{sL, sR}(\tilde{A}), r\mathcal{V}_A(\tilde{A}) + (1 - r)A_{sL, sR}(\tilde{A}) - A_{sL, sR}(\tilde{A}))_\lambda = r(\mathcal{V}_A(\tilde{A}) - A_{sL, sR}(\tilde{A}), \mathcal{V}_A(\tilde{A}) - A_{sL, sR}(\tilde{A}))_\lambda > 0,
\]

which is a contradiction. Hence \( x' = 0 \) and then the weighted \( (s_L, s_R) \) semi-triangular approximation \( A_{sL, sR}(\tilde{A}) \) must be of the form

\[
A_{sL, sR}(\tilde{A}) = c'(\tilde{E}_1 + \tilde{E}_3) + y'\tilde{E}_4.
\]

Fact 2.3 implies

\[
(c', y') = ((\tilde{A}, \tilde{E}_1 + \tilde{E}_3), (\tilde{A}, \tilde{E}_4))_\lambda \left( \begin{array}{cc} \Phi_{11} + \Phi_{33} & \Phi_{34} \\ \Phi_{34} & \Phi_{44} \end{array} \right)^{-1}.
\]

It needs to prove \( y' \geq 0 \). Indeed, (4.5) and (4.8) implies

\[
\left| \begin{array}{cc} \Phi_{11} + \Phi_{33} & \Phi_{34} \\ \Phi_{34} & \Phi_{44} \end{array} \right| = \Phi_{11}\Phi_{44} + \Phi_{33}\Phi_{44} > 0,
\]

and from Lemma 5.2 we find

\[
y' = \left| \begin{array}{cc} \Phi_{11} + \Phi_{33} & \Phi_{34} \\ \Phi_{11} + \Phi_{33} & \Phi_{44} \end{array} \right| \geq 0.
\]

This proves Case (2). In the same way, we prove Case (3). \( \square \)
Let $V_E(\tilde{A}) = \sum_{i=1}^4 x_i \tilde{E}_i$ be the projection of $\tilde{A}$ onto $\bigvee E$. In Theorem 4.4 we have shown that $T_{s_L,s_R}(\tilde{A}) = V_E(\tilde{A})$ iff $x_1 - x_3 \leq 0$. By (4.9) and Cramer’s Rule we obtain

$$x_1 - x_3 = \frac{1}{\det \Phi} \begin{vmatrix} \langle \tilde{A}, \tilde{E}_1 \rangle & \langle \tilde{A}, \tilde{E}_2 \rangle & \langle \tilde{A}, \tilde{E}_3 \rangle & \langle \tilde{A}, \tilde{E}_4 \rangle \\ \Phi_{12} & \Phi_{22} & \Phi_{33} & \Phi_{34} \\ \Phi_{12} & \Phi_{22} & \Phi_{34} & \Phi_{44} \\ \Phi_{12} & \Phi_{34} & \Phi_{34} & \Phi_{44} \end{vmatrix} - \frac{1}{\det \Phi} \begin{vmatrix} \langle \tilde{A}, \tilde{E}_1 \rangle & \langle \tilde{A}, \tilde{E}_2 \rangle & \langle \tilde{A}, \tilde{E}_3 \rangle & \langle \tilde{A}, \tilde{E}_4 \rangle \\ \Phi_{11} & \Phi_{12} & 0 & 0 \\ \Phi_{12} & \Phi_{22} & 0 & 0 \\ \Phi_{12} & \Phi_{34} & \Phi_{34} & \Phi_{44} \end{vmatrix}.$$ 

Define

$$\Gamma_0 := \left\{ \tilde{A} \in F(\mathbb{R}) | \begin{vmatrix} \langle \tilde{A}, \tilde{E}_1 \rangle & \langle \tilde{A}, \tilde{E}_2 \rangle & \langle \tilde{A}, \tilde{E}_3 \rangle & \langle \tilde{A}, \tilde{E}_4 \rangle \\ \Phi_{11} & \Phi_{12} & 0 & 0 \\ \Phi_{12} & \Phi_{22} & 0 & 0 \\ \Phi_{12} & \Phi_{34} & \Phi_{34} & \Phi_{44} \end{vmatrix} \leq 0 \right\}. \quad (5.13)$$

**Lemma 5.6.** $\Gamma_0 \subseteq \Gamma_1$.

**Proof.** First, we show that $\Gamma_0 \cap \Gamma_2 = \emptyset$. From (5.13) we find $\tilde{A} \in \Gamma_0$ iff

$$0 \leq \Phi_{44} \langle \tilde{A}, \tilde{E}_3 \rangle - \Phi_{34} \langle \tilde{A}, \tilde{E}_4 \rangle + \frac{\Phi_{33} \Phi_{44} - \Phi_{34}^2}{\Phi_{11} \Phi_{22} - \Phi_{12}^2} (\Phi_{12} \langle \tilde{A}, \tilde{E}_2 \rangle - \Phi_{22} \langle \tilde{A}, \tilde{E}_1 \rangle) \quad (5.14)$$

and from (5.3) $\tilde{A} \in \Gamma_2$ iff

$$0 < \Phi_{12} (\Phi_{34} \langle \tilde{A}, \tilde{E}_4 \rangle - \Phi_{44} \langle \tilde{A}, \tilde{E}_3 \rangle) + \frac{\Phi_{33} \Phi_{44} - \Phi_{34}^2}{\Phi_{11} \Phi_{22} - \Phi_{12}^2} (\Phi_{12} \langle \tilde{A}, \tilde{E}_2 \rangle - \Phi_{22} \langle \tilde{A}, \tilde{E}_1 \rangle) + \Phi_{44} \left| \begin{array}{ll} \Phi_{11} & \Phi_{12} \\ \langle \tilde{A}, \tilde{E}_1 \rangle & \langle \tilde{A}, \tilde{E}_2 \rangle \end{array} \right|. \quad (5.15)$$

Suppose on the contrary that $\Gamma_0 \cap \Gamma_2 \neq \emptyset$ and let $\tilde{A} \in \Gamma_0 \cap \Gamma_2$. By computing the sum of (5.14) multiplied by $\Phi_{12}$ and (5.15) we get

$$0 < \Phi_{33} \Phi_{44} - \Phi_{34}^2 \left( \Phi_{12} \langle \tilde{A}, \tilde{E}_2 \rangle - \Phi_{22} \langle \tilde{A}, \tilde{E}_1 \rangle \right) \Phi_{12} + \frac{\Phi_{33} \Phi_{44} - \Phi_{34}^2}{\Phi_{11} \Phi_{22} - \Phi_{12}^2} (\Phi_{12} \langle \tilde{A}, \tilde{E}_2 \rangle - \Phi_{22} \langle \tilde{A}, \tilde{E}_1 \rangle) + \Phi_{44} \left| \begin{array}{ll} \Phi_{11} & \Phi_{12} \\ \langle \tilde{A}, \tilde{E}_1 \rangle & \langle \tilde{A}, \tilde{E}_2 \rangle \end{array} \right|. \quad (5.16)$$

By (4.7), (4.8) and $\Phi_{22}, \Phi_{44} > 0$, the above inequality contradicts (4.10).

In the same way, we prove $\Gamma_0 \cap \Gamma_3 = \emptyset$. By Lemma 5.3 we obtain $\Gamma_0 \subseteq \Gamma_1$. \[ \square \]

Combining Theorem 4.4 with Theorem 5.5 we obtain the following theorem.

**Theorem 5.7.** Let $\tilde{A} \in F(\mathbb{R})$. Then the weighted $(s_L, s_R)$ semi-trapezoidal approximation $T_{s_L,s_R}(\tilde{A})$ of $\tilde{A}$ can be computed in the following cases:

1. If $\tilde{A} \in \Gamma_0$, then $T_{s_L,s_R}(\tilde{A}) = \lfloor x_1 + x_2(1 - t)^{1/s_L}, x_3 + x_4(1 - t)^{1/s_R} \rfloor$, where

$$\begin{pmatrix} \Phi_{11} & \Phi_{12} & 0 & 0 \\ \Phi_{12} & \Phi_{22} & 0 & 0 \\ 0 & 0 & \Phi_{33} & \Phi_{34} \\ 0 & 0 & \Phi_{34} & \Phi_{44} \end{pmatrix}^{-1} \begin{pmatrix} \langle \tilde{A}, \tilde{E}_1 \rangle & \langle \tilde{A}, \tilde{E}_2 \rangle & \langle \tilde{A}, \tilde{E}_3 \rangle & \langle \tilde{A}, \tilde{E}_4 \rangle \end{pmatrix}.$$
(2) If \( \tilde{A} \in \Gamma_1 \setminus \Gamma_0 \), then \( T_{s_L,s_R}(\tilde{A}) = [c + \epsilon(1 - t)^{1/s_L}, c + \rho(1 - t)^{1/s_R}] \), where

\[
(c, \epsilon, \rho) = ((\tilde{\epsilon}, \tilde{E}_1 + \tilde{E}_3), (\tilde{\rho}, \tilde{E}_2), (\tilde{\epsilon}, \tilde{E}_4)) \left( \begin{array}{ccc}
\Phi_{11} + \Phi_{33} & \Phi_{12} & \Phi_{34} \\
\Phi_{34} & 0 & 0 \\
0 & 0 & 0
\end{array} \right)^{-1}.
\]

(3) If \( \tilde{A} \in \Gamma_2 \), then \( T_{s_L,s_R}(\tilde{A}) = [c', c' + \epsilon'(1 - t)^{1/s_R}] \), where

\[
(c', \epsilon') = (\tilde{\epsilon}, \tilde{E}_1 + \tilde{E}_3, (\tilde{\epsilon}, \tilde{E}_4)) \left( \begin{array}{ccc}
\Phi_{11} + \Phi_{33} & \Phi_{12} & \Phi_{34} \\
\Phi_{34} & 0 & 0 \\
0 & 0 & 0
\end{array} \right)^{-1}.
\]

(4) If \( \tilde{A} \in \Gamma_3 \), then \( T_{s_L,s_R}(\tilde{A}) = [c', \epsilon'(1 - t)^{1/s_L}, c'] \), where

\[
(c', x') = (\tilde{\epsilon}, \tilde{E}_1 + \tilde{E}_3, (\tilde{\epsilon}, \tilde{E}_4)) \left( \begin{array}{ccc}
\Phi_{11} + \Phi_{33} & \Phi_{12} & \Phi_{34} \\
\Phi_{34} & 0 & 0 \\
0 & 0 & 0
\end{array} \right)^{-1}.
\]

6. Examples

Theorem 5.7 is a generalization of the two papers: Ban [8] and Yeh [29]. Indeed, when setting \( \hat{\lambda}_L(t) = \hat{\lambda}_U(t) = 1 \), Theorem 5.7 represents Ban’s formula [8, Theorem 4] of matrix type for \((s_L, s_R)\)-semi-trapezoidal approximations, and when setting \( s_L = s_R = 1 \) it represents Yeh’s formula [29, Theorem 5.2] for weighted trapezoidal approximations.

Now, let us fix the following objectives:

\[
\begin{align*}
\Phi_{11} &= \int_0^1 \lambda_L(t) \, dt, & \Phi_{12} &= \int_0^1 (1 - t)^{1/s_L} \hat{\lambda}_L(t) \, dt, & \Phi_{22} &= \int_0^1 (1 - t)^{2/s_L} \hat{\lambda}_L(t) \, dt, \\
\Phi_{33} &= \int_0^1 \lambda_U(t) \, dt, & \Phi_{34} &= \int_0^1 (1 - t)^{1/s_R} \hat{\lambda}_U(t) \, dt, & \Phi_{44} &= \int_0^1 (1 - t)^{2/s_R} \hat{\lambda}_U(t) \, dt,
\end{align*}
\]

\[
\Phi = \begin{pmatrix}
\Phi_{11} & \Phi_{12} & 0 & 0 \\
\Phi_{12} & \Phi_{22} & 0 & 0 \\
0 & 0 & \Phi_{33} & \Phi_{34} \\
0 & 0 & \Phi_{34} & \Phi_{44}
\end{pmatrix},
\]

and \( r_i := (\tilde{\epsilon}, \tilde{E}_i), 1 \leq i \leq 4 \), that is

\[
\begin{align*}
r_1 &= \int_0^1 A_L(t) \lambda_L(t) \, dt, & r_2 &= \int_0^1 A_L(t)(1 - t)^{1/s_L} \hat{\lambda}_L(t) \, dt, \\
r_3 &= \int_0^1 A_U(t) \lambda_U(t) \, dt, & r_4 &= \int_0^1 A_U(t)(1 - t)^{1/s_R} \hat{\lambda}_U(t) \, dt.
\end{align*}
\]

Example 6.1 (Ban [8, Example 6]). Let \( \lambda_L(t) = \lambda_U(t) = 1 \) and \( s_L = s_R = 2 \). Consider the weighted \((2, 2)\)-semi-trapezoidal approximation of fuzzy number \( \tilde{A} = [1 + \sqrt{t}, 30 - 27\sqrt{t}], t \in [0, 1] \).

First, compute the following objectives:

\[
\begin{align*}
\Phi_{11} &= \Phi_{33} = \int_0^1 dt = 1, \\
\Phi_{12} &= \Phi_{34} = \int_0^1 (1 - t)^{1/2} \, dt = \frac{2}{3}.
\end{align*}
\]
Since \( \int_0^1 t^{1/2} (1-t)^{1/2} \, dt = \pi/8 \) we obtain

\[
\begin{align*}
r_1 &= \int_0^1 (1 + \sqrt{t}) \, dt = \frac{5}{3}, \\
r_2 &= \int_0^1 (1 + \sqrt{t})(1-t)^{1/2} \, dt = \frac{2}{3} + \frac{1}{8} \pi, \\
r_3 &= \int_0^1 (30 - 27\sqrt{t}) \, dt = 12, \\
r_4 &= \int_0^1 (30 - 27\sqrt{t})(1-t)^{1/2} \, dt = 20 - \frac{27}{8} \pi.
\end{align*}
\]

It is easy to verify that \( \bar{A} \in T_2 \). Hence, by Theorem 5.7 case (3) we compute

\[
(c', y') = \left( \frac{5}{3} + 12, 20 - \frac{27}{8} \pi \right) \left( \frac{2}{3} + \frac{2}{3} \pi \right)^{-1} = \left( \frac{-117}{10} + \frac{81}{20} \pi, \frac{278}{5} - \frac{243}{20} \pi \right),
\]

and then the weighted (2, 2) semi-trapezoidal approximation of \( \bar{A} \) is

\[
T_{2,2}(\bar{A}) = \left[ \frac{-117}{10} + \frac{81}{20} \pi, \frac{-117}{10} + \frac{81}{20} \pi + \left( \frac{278}{5} - \frac{243}{20} \pi \right) (1-t)^{1/2} \right].
\]

**Example 6.2 (Yeh \cite[Example 6.2]{Yeh1996}).** Let \( \hat{x}_L(t) = 1 - t, \hat{x}_U(t) = t, \) and \( s_L = s_R = 1 \). Consider the weighted \((1, 1)\) semi-trapezoidal approximation of fuzzy number \( \bar{A} = [-1 + \sqrt{t}, 1 - t^2], t \in [0, 1] \).

Throughout mathematical computation, we obtain the following objectives:

\[
\Phi_{11} = \frac{1}{7}, \quad \Phi_{12} = \frac{1}{5}, \quad \Phi_{22} = \frac{1}{4}, \quad \Phi_{33} = \frac{1}{5}, \quad \Phi_{34} = \frac{1}{6}, \quad \Phi_{44} = \frac{1}{12},
\]

and \( r_1 = -7/30, r_2 = -19/105, r_3 = 1/4, r_4 = 7/60 \). Consequently, substituting these objectives into (5.2) and (5.13) we find \( \bar{A} \in T_1 \setminus T_0 \). Hence by Theorem 5.7 case (2) we get

\[
(c, x, y) = \left( -\frac{7}{30} + \frac{1}{4}, -\frac{19}{105}, \frac{7}{60} \right) \left( \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, \frac{1}{5}, \frac{1}{5} \right)^{-1} = \left( \frac{31}{280}, -\frac{61}{70}, \frac{33}{28} \right).
\]

Thus, the weighted \((1, 1)\) semi-trapezoidal approximation of \( \bar{A} \) is

\[
T_{1,1}(\bar{A}) = \left[ \frac{31}{280} - \frac{61}{70} (1-t), \frac{31}{280} + \frac{33}{28} (1-t) \right].
\]

**Example 6.3.** Let \( \hat{x}_L(t) = 1 - t, \hat{x}_U(t) = t \) and \( s_L = 1, s_R = 12 \). Consider the weighted \((1, 12)\) semi-trapezoidal approximation and weighted \((1, 12)\) semi-triangular approximation of fuzzy number \( \bar{A} = [-1 + \sqrt{t}, 1 - t^2], t \in [0, 1] \).
Throughout mathematical computation, we obtain

\[
\Phi = \begin{pmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\
\Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} \\
\Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} \\
\Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{3} & 0 & 0 \\
\frac{1}{3} & \frac{1}{4} & 0 & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{12} \\
0 & 0 & \frac{1}{12} & \frac{1}{30}
\end{pmatrix}
\]

and \( r_1 = -7/30, r_2 = -19/105, r_3 = 1/4, r_4 = 1/15 \). Substituting into (5.13) we find \( \tilde{A} \in \Gamma_0 \). By Theorem 5.7 case (1) we get

\[
(x_1, x_2, x_3, x_4) = \left( -\frac{7}{30}, -\frac{19}{105}, \frac{1}{4}, \frac{1}{15} \right) \Phi^{-1} = \left( \frac{1}{7}, -\frac{32}{35}, \frac{2}{7}, \frac{9}{7} \right).
\]

Hence, the weighted (1, 1/2) semi-trapezoidal approximation of \( \tilde{A} \) is

\[
T_{1,1/2}(\tilde{A}) = \left[ \frac{1}{7} - \frac{32}{35} (1-t), \frac{2}{7} + \frac{9}{7} (1-t)^2 \right].
\]

Now, we compute the weighted (1, 1/2) semi-triangular approximation of \( \tilde{A} \). From Lemma 5.6 we find \( \tilde{A} \in \Gamma_0 \subseteq \Gamma_1 \). By Theorem 5.5 case (1) we get

\[
(c, x, y) = \left( -\frac{7}{30} + \frac{1}{4}, -\frac{19}{105}, \frac{1}{15} \right) \left( \frac{1}{3}, \frac{1}{4}, 0, \frac{1}{12}, 0, \frac{1}{12}, 0, \frac{1}{30} \right)^{-1} = \left( \frac{46}{175}, -\frac{188}{175}, \frac{47}{35} \right).
\]

Therefore, the weighted (1, 1/2) semi-triangular approximation of \( \tilde{A} \) is

\[
A_{1,1/2}(\tilde{A}) = \left[ \frac{46}{175} - \frac{188}{175} (1-t), \frac{46}{175} + \frac{47}{35} (1-t)^2 \right].
\]

7. Properties

Grzegorzewski and Mrówka [16] proposed many properties of trapezoidal approximations. Here we investigate properties of weighted \((s_L, s_R)\) semi-trapezoidal approximations and weighted \((s_L, s_R)\) semi-triangular approximations.

**Proposition 7.1.** Weighted \((s_L, s_R)\) semi-trapezoidal approximations and weighted \((s_L, s_R)\) semi-triangular approximations both satisfy the identity criterion.

**Proof.** From Section 4, we find that the weighted \((s_L, s_R)\) semi-trapezoidal approximation \( T_{s_L,s_R}(\tilde{A}) \) of \( \tilde{A} \) is the best approximation of \( \tilde{A} \) in the closed convex subset \( \mathbb{T}_{s_L,s_R} \). Thus, if \( \tilde{A} \in \mathbb{T}_{s_L,s_R} \), then \( T_{s_L,s_R}(\tilde{A}) = \tilde{A} \). This proves that \( T_{s_L,s_R} \) satisfies the identity criterion. In addition, from Section 5 we find that the weighted \((s_L, s_R)\) semi-triangular approximation \( A_{s_L,s_R}(\tilde{A}) \) of \( \tilde{A} \) is the best approximation of \( \tilde{A} \) in the closed convex subset \( \mathbb{T}_{s_L,s_R}^A \). Hence, if \( \tilde{A} \in \mathbb{T}_{s_L,s_R}^A \), then \( A_{s_L,s_R}(\tilde{A}) = \tilde{A} \). This completes the proof. \( \square \)

Recall that, a function \( f = f(x) \) is Lipschitz continuous if it satisfies

\[
d(f(x), f(y)) \leq d(x, y).
\]

From Fact 2.2, we find that every best approximation in a closed convex subset is Lipschitz continuous. Hence, we prove:

**Proposition 7.2.** Weighted \((s_L, s_R)\) semi-trapezoidal approximations and weighted \((s_L, s_R)\) semi-triangular approximations are both Lipschitz continuous.
Recall that, an operator $P$ is invariant to translations if
$$P(\tilde{A} + z) = P(\tilde{A}) + z, \quad \forall \tilde{A} \in F(\mathbb{R}), z \in \mathbb{R}.$$ 

**Proposition 7.3.** Weighted $(s_L, s_R)$ semi-trapezoidal approximations and weighted $(s_L, s_R)$ semi-triangular approximations are both invariant to translations.

**Proof.** Let $T_{s_L, s_R}(\tilde{A})$ be the weighted $(s_L, s_R)$ semi-trapezoidal approximation of $\tilde{A}$, and $z \in \mathbb{R}$. By Fact 2.2 we get
$$\langle \tilde{A} - T_{s_L, s_R}(\tilde{A}), \tilde{X} - T_{s_L, s_R}(\tilde{A}) \rangle_{\lambda} \leq 0, \quad \forall \tilde{X} \in \mathbb{T}_{s_L, s_R}.$$

Hence we obtain
$$((\tilde{A} + z) - (T_{s_L, s_R}(\tilde{A}) + z), \tilde{X} - (T_{s_L, s_R}(\tilde{A}) + z))_{\lambda} = ((\tilde{A} - T_{s_L, s_R}(\tilde{A})), (\tilde{X} - z) - T_{s_L, s_R}(\tilde{A}))_{\lambda} \leq 0.$$

By applying Fact 2.2 again, the weighted $(s_L, s_R)$ semi-trapezoidal approximation of $\tilde{A} + z$ is $T_{s_L, s_R}(\tilde{A}) + z$, i.e.
$$T_{s_L, s_R}(\tilde{A} + z) = T_{s_L, s_R}(\tilde{A}) + z.$$

In the same way, we prove the $(s_L, s_R)$ semi-triangular approximation is invariant to translations, too. $\square$

Let $\tilde{A}, \tilde{B} \in F(\mathbb{R})$. In Section 3, we have shown that
$$r \odot \tilde{A} \oplus s \odot \tilde{B} = r \tilde{A} + s \tilde{B}, \quad \forall r, s \geq 0,$$

where “$\odot$” and “$\oplus$” are fuzzy operations. In each case of Theorem 5.7, $T_{s_L, s_R}(\tilde{A})$ is always a projection onto a subspace, and so is $A_{s_L, s_R}$ in each case of Theorem 5.5. Thus, they are linear with respect to vector operations. So, we prove:

**Proposition 7.4.**

1. Let $\tilde{A}$ and $\tilde{B}$ belong to the same case in Theorem 5.7. Then
$$T_{s_L, s_R}(r \odot \tilde{A} \oplus s \odot \tilde{B}) = r \odot T_{s_L, s_R}(\tilde{A}) \oplus s \odot T_{s_L, s_R}(\tilde{B}), \quad \forall r, s \geq 0.$$

2. Let $\tilde{A}$ and $\tilde{B}$ belong to the same case in Theorem 5.5. Then
$$A_{s_L, s_R}(r \odot \tilde{A} \oplus s \odot \tilde{B}) = r \odot A_{s_L, s_R}(\tilde{A}) \oplus s \odot A_{s_L, s_R}(\tilde{B}), \quad \forall r, s \geq 0.$$

From Proposition 7.4, by setting $s = 0$ we find
$$T_{s_L, s_R}(r \odot \tilde{A}) = r \odot T_{s_L, s_R}(\tilde{A}) \quad \text{and} \quad A_{s_L, s_R}(r \odot \tilde{A}) = r \odot A_{s_L, s_R}(\tilde{A}), \quad \forall r \geq 0. \quad (7.1)$$

In the following, we will show a similar result for the case of $r < 0$. For avoiding confusion we have to use more detailed notations. Given weighted functions $\lambda_L = \lambda_L(t), \lambda_U = \lambda_U(t)$ and $s_L, s_R > 0$, let $T_{s_L, s_R}^{\lambda_L, \lambda_U}(\tilde{A})$ and $A_{s_L, s_R}^{\lambda_L, \lambda_U}(\tilde{A})$ denote the weighted $(s_L, s_R)$ semi-trapezoidal and weighted $(s_L, s_R)$ semi-triangular approximations of $\tilde{A}$ respectively, $(\tilde{A}, \tilde{B})_{\lambda_L, \lambda_U}$ denote the inner product defined by (3.2), and $\tilde{E}_{s_L, s_R}(i)$ denote the detailed notation of $\tilde{E}_t$ defined by (4.1). Further
$$\phi_{s_L, s_R}^{\lambda_L, \lambda_U}(i, j) := (\tilde{E}_{s_L, s_R}(i), \tilde{E}_{s_L, s_R}(j))_{\lambda_L, \lambda_U}, \quad 0 \leq i, j \leq 4,$$
$$\phi_{s_L, s_R}^{\lambda_L, \lambda_U} := [\phi_{s_L, s_R}^{\lambda_L, \lambda_U}(i, j)].$$

Note that
$$\phi_{s_L, s_R}^{\lambda_L, \lambda_U}(i, j) = \phi_{s_R, s_L}^{\lambda_U, \lambda_L}(i + 2, j + 2), \quad i = 1, 2, \quad j = 1, 2 \quad \text{and} \quad (7.2)$$
and
$$\langle \Theta \tilde{A}, \tilde{E}_{s_L, s_R}(i) \rangle_{\lambda_L, \lambda_U} = \begin{cases} -(\tilde{A}, \tilde{E}_{s_R, s_L}(i + 2))_{\lambda_U, \lambda_L}, & i = 1, 2, \\ -(\tilde{A}, \tilde{E}_{s_R, s_L}(i - 2))_{\lambda_U, \lambda_L}, & i = 3, 4. \end{cases} \quad (7.3)$$

where $\Theta \tilde{A} := (\tilde{A}, \tilde{A})$. 


Proposition 7.5. Let $\tilde{A} \in F$. Then for all $r \leq 0$ we have
\[
T_{s_l^r, s_r^l}^h (r \odot \tilde{A}) = r \odot T_{s_r^l, s_l^r}^h (\tilde{A})
\] (7.4)
and
\[
A_{s_l^r, s_r^l}^h (r \odot \tilde{A}) = r \odot A_{s_r^l, s_l^r}^h (\tilde{A}).
\] (7.5)

Proof. By (7.1) it suffices to prove
\[
T_{s_l^r, s_r^l}^h (\ominus \tilde{A}) = \ominus T_{s_r^l, s_l^r}^h (\tilde{A})
\] and
\[
A_{s_l^r, s_r^l}^h (\ominus \tilde{A}) = \ominus A_{s_r^l, s_l^r}^h (\tilde{A}).
\]
Similarly, let $\Gamma_{s_l^r, s_r^l}^h (i)$ denote the detailed notation of $\Gamma_i$, $0 \leq i \leq 3$. First, assume $\ominus \tilde{A} \in \Gamma_{s_l^r, s_r^l}^h (0)$. Eqs. (7.2) and (7.3) together implies
\[
\begin{align*}
(\ominus \tilde{A}, \tilde{E}_{s_r^l, s_l^r}(1))_{s_l^r, s_r^l} & \Phi_{s_l^r, s_r^l}^h (2, 2) - (\ominus \tilde{A}, \tilde{E}_{s_r^l, s_l^r}(2))_{s_l^r, s_r^l} \Phi_{s_l^r, s_r^l}^h (1, 2) \\
& = - \frac{\Phi_{s_l^r, s_r^l}^h (1, 1) \Phi_{s_l^r, s_r^l}^h (2, 2) - \Phi_{s_l^r, s_r^l}^h (1, 2)^2}{\Phi_{s_l^r, s_r^l}^h (3, 3) \Phi_{s_l^r, s_r^l}^h (4, 4) - \Phi_{s_l^r, s_r^l}^h (3, 4)^2}
\end{align*}
\]
and
\[
\begin{align*}
(\ominus \tilde{A}, \tilde{E}_{s_r^l, s_l^r}(3))_{s_l^r, s_r^l} & \Phi_{s_l^r, s_r^l}^h (2, 2) - (\ominus \tilde{A}, \tilde{E}_{s_r^l, s_l^r}(2))_{s_l^r, s_r^l} \Phi_{s_l^r, s_r^l}^h (1, 2) \\
& = - \frac{\Phi_{s_l^r, s_r^l}^h (1, 1) \Phi_{s_l^r, s_r^l}^h (2, 2) - \Phi_{s_l^r, s_r^l}^h (1, 2)^2}{\Phi_{s_l^r, s_r^l}^h (3, 3) \Phi_{s_l^r, s_r^l}^h (4, 4) - \Phi_{s_l^r, s_r^l}^h (3, 4)^2}
\end{align*}
\]
Hence by (5.13) we find $\tilde{A} \in \Gamma_{s_l^r, s_r^l}^h (0)$. Let $T_{s_r^l, s_l^r}^h (\ominus \tilde{A}) = \sum_{i=1}^{4} x_i \tilde{E}_{s_r^l, s_l^r}$ and $T_{s_l^r, s_r^l}^h (\tilde{A}) = \sum_{i=1}^{4} y_i \tilde{E}_{s_r^l, s_l^r}$. By (7.2), (7.3) and Theorem 5.7 case (1) we obtain
\[
(x_1, x_2) = ((\ominus \tilde{A}, E_{s_r^l, s_l^r}(1))_{s_l^r, s_r^l}, (\ominus \tilde{A}, E_{s_r^l, s_l^r}(2))_{s_l^r, s_r^l}) \left( \begin{array}{cc} \Phi_{s_l^r, s_r^l}^h (1, 1) & \Phi_{s_l^r, s_r^l}^h (1, 2) \\ \Phi_{s_l^r, s_r^l}^h (1, 2) & \Phi_{s_l^r, s_r^l}^h (2, 2) \end{array} \right)^{-1}
\]
and
\[
(x_3, x_4) = ((\ominus \tilde{A}, E_{s_r^l, s_l^r}(3))_{s_l^r, s_r^l}, (\ominus \tilde{A}, E_{s_r^l, s_l^r}(4))_{s_l^r, s_r^l}) \left( \begin{array}{cc} \Phi_{s_l^r, s_r^l}^h (3, 3) & \Phi_{s_l^r, s_r^l}^h (3, 4) \\ \Phi_{s_l^r, s_r^l}^h (3, 4) & \Phi_{s_l^r, s_r^l}^h (4, 4) \end{array} \right)^{-1}
\]
Hence,
\[
\sum_{i=1}^{4} y_i \tilde{E}_{s_r^l, s_l^r} = (y_1 + y_2 (1-t)^{1/s_r}, y_3 + y_4 (1-t)^{1/s_l}) \left[ -y_3 - y_4 (1-t)^{1/s_l}, -y_1 - y_2 (1-t)^{1/s_r} \right]
\]
In the same way, we prove (7.5). □
Corollary 7.6. Suppose that $\lambda_L(t) = \lambda_U(t)$ and $s_L = s_R = s$.
(1) If $\bar{A}$ and $\bar{B}$ belong to the same case in Theorem 5.7, then
\[ T_{s,s}(a \odot A \oplus b \odot \bar{B}) = a \odot T_{s,s}(\bar{A}) \oplus b \odot T_{s,s}(\bar{B}), \quad \forall a, b \in \mathbb{R}. \]
(2) If $\bar{A}$ and $\bar{B}$ belong to the same case in Theorem 5.5, then
\[ A_{s,s}(a \odot \bar{A} \oplus b \odot \bar{B}) = a \odot A_{s,s}(\bar{A}) \oplus b \odot A_{s,s}(\bar{B}), \quad \forall a, b \in \mathbb{R}. \]

8. Conclusions

In 2002, Grzegorzewski [15] first applied Lagrange multiplier to solve the problem of approximations of fuzzy numbers. Later, Ban [7] proposed a new method by applying Karush–Kuhn–Tucker Theorem [24, pp. 281–283]. In fact, Karush–Kuhn–Tucker Theorem is a generalization of Lagrange multiplier. In the present paper, we propose another method by applying approximation theory in Hilbert spaces. The proposed formulas are presented by matrix type. It makes computation of \((s_L, s_R)\) semi-trapezoidal approximations quite simple and clear. Up to now, the linear shaped approximations of fuzzy numbers have been solved completely. In the future, we should follow Grzegorzewski and Stefanini’s investigation [19] to study non-linear shaped approximations. Actually, our method can be applied to solve non-linear shaped approximations. Suppose that we are going to find the best approximation of $A$ in a closed convex subset $\Omega$. We describe our method as the following steps:

Step 1: Find the minimal subspace $V$ that contains $\Omega$.
Step 2: Find a basis for $V$.
Step 3: Compute the projection of $\bar{A}$ onto the subspace $V$ which is denoted by $P_V(\bar{A})$.
Step 4: Check whether $P_V(\bar{A})$ belongs to $\Omega$ or not. If yes, then the best approximation of $\bar{A}$ is $P_V(\bar{A})$. Otherwise, we will get a new closed convex subset $\Omega'$ properly contained in $\Omega$. By Fact 2.4, we only consider the reduced problem: the best approximation of $\bar{A}$ in $\Omega'$. Then, solve the reduced problem by Steps 1–4 once again.

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Appendix

Theorem (Chebyshev’s Inequality). Let $\lambda = \lambda(t)$ be a weighted function on $[0, 1]$, and let $f = f(t)$ and $g = g(t)$ be both increasing functions on $[0, 1]$. Then
\[ \int_{0}^{1} f(t)g(t)\lambda(t)\,dt \cdot \int_{0}^{1} \lambda(t)\,dt \geq \int_{0}^{1} f(t)\lambda(t)\,dt \cdot \int_{0}^{1} g(t)\lambda(t)\,dt. \]

Proof. Let $G = G(t)$ denote the function
\[ G(t) = \int_{0}^{t} [g(x) - w]\lambda(x)\,dx \quad \text{where} \quad w = \frac{\int_{0}^{1} g(x)\lambda(x)\,dx}{\int_{0}^{1} \lambda(x)\,dx}. \]
Obviously, $G(0) = F(1) = 0$. In addition, fundamental theorem of calculus implies
\[ G'(t) = [g(t) - w]\lambda(t). \]
Since $g = g(t)$ is increasing and $\lambda = \lambda(t)$ is nonnegative, there exists a $c \in [0, 1]$ such that $G'(t) < 0$ if $t \in [0, c)$ and $G'(t) > 0$ if $t \in (c, 1]$. This shows that $G = G(t)$ is decreasing on $[0, c]$ and increasing on $[c, 1]$. Hence,
\[ G(t) \leq 0, \quad \forall t \in [0, 1]. \]
By using integration by parts, we compute the integral
\[
\int_0^1 f(t)(g(t) - w)\dot{\lambda}(t)\,dt = f(t)G(t)|_0^1 - \int_0^1 G(t)\,df(t) = -\int_0^1 G(t)\,df(t).
\]
The hypothesis, \(f = f(t)\) is increasing, implies \(df(t) \geq 0\), hence
\[
\int_0^1 f(t)(g(t) - w)\dot{\lambda}(t)\,dt \geq 0,
\]
that is
\[
\int_0^1 f(t)g(t)\dot{\lambda}(t)\,dt \geq w\int_0^1 f(t)\dot{\lambda}(t)\,dt = \frac{\int_0^1 g(t)\dot{\lambda}(t)\,dt}{\int_0^1 \dot{\lambda}(t)\,dt} \cdot \int_0^1 f(t)\dot{\lambda}(t)\,dt.
\]
This completes the proof. \(\square\)

References