Weighted trapezoidal and triangular approximations of fuzzy numbers

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Abstract
In 2007, Zeng and Li proposed a weighted triangular approximation of a fuzzy number. Unfortunately, this approximation may fail to be a fuzzy number. In this paper, we improve this approximation and propose a generalization by the name of weighted trapezoidal approximation. Their algorithms are also presented. Finally, some examples and relevant properties are discussed.

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1. Introduction
An arbitrary fuzzy number \( \tilde{A} \) can be represented by an ordered pair of left continuous functions \( [A_L(x), A_U(x)] \), \( 0 \leq x \leq 1 \), which satisfy the following conditions: (1) \( A_L \) is increasing on \([0,1]\), (2) \( A_U \) is decreasing on \([0,1]\), (3) \( A_L(1) \leq A_U(1) \). Let \( \mathbb{F} \) and \( \mathbb{R} \) denote the sets of all fuzzy numbers and real numbers, respectively. A function \( d: \mathbb{F} \times \mathbb{F} \to \mathbb{R} \) is called a distance on fuzzy numbers if it obeys

1. \( d(\tilde{A}, \tilde{B}) \geq 0 \) and \( d(\tilde{A}, \tilde{B}) = 0 \) iff (if and only if) \( \tilde{A} = \tilde{B} \),
2. \( d(\tilde{A}, \tilde{B}) = d(\tilde{B}, \tilde{A}) \),
3. \( d(\tilde{A}, \tilde{C}) \leq d(\tilde{A}, \tilde{B}) + d(\tilde{B}, \tilde{C}) \),

for \( \tilde{A}, \tilde{B}, \tilde{C} \in \mathbb{F} \). In [8], Grzegorzewski proposed two families of distances on \( \mathbb{F} \): \( \delta_{p,q} \) and \( \rho_p \). In which, if \( 1 \leq p < \infty \), \( \delta_{p,q} \) is defined as follows:

\[
\delta_{p,q}(\tilde{A}, \tilde{B}) := \left[ (1-q) \int_0^1 |A_L(x) - B_L(x)|^p \, dx + q \int_0^1 |A_U(x) - B_U(x)|^p \, dx \right]^{1/p},
\]

(1.1)

where \( 0 < q < 1 \). The famous \( L_2 \)-distance on \( \mathbb{F} \) is defined as

\[
d(\tilde{A}, \tilde{B}) = \left[ \int_0^1 (|A_L(x) - B_L(x)|^2 + |A_U(x) - B_U(x)|^2) \, dx \right]^{1/2},
\]

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that is \( d(\tilde{A}, \tilde{B}) = \sqrt{2} \cdot \delta_{2,1/2}(\tilde{A}, \tilde{B}) \). Recall that, a fuzzy number \( \tilde{A} \) is called trapezoidal if its \( x \)-cuts are of the form \([t_i + (t_i - t_1)x, t_3 - (t_1 - t_i)x] \), where \( t_i, 1 \leq i \leq 4 \), are real numbers with the constraint:

\[
t_1 \leq t_2 \leq t_3 \leq t_4.
\]

Furthermore, a trapezoidal fuzzy number is called triangular (or rectangle) if \( t_2 = t_3 \) (or \( t_1 = t_2 = t_3 = t_4 \), respectively) holds.

Let \( d(\cdot, \cdot) \) be an arbitrary distance on \( \mathbb{F} \), and \( \tilde{A} \) be a fuzzy number. The trapezoidal (triangular, symmetric triangular, interval) approximation of \( \tilde{A} \), under the distance \( d(\cdot, \cdot) \), is the trapezoidal (triangular, symmetric triangular, rectangle) fuzzy number which minimizes the distance \( d(\tilde{A}, \tilde{X}) \), where \( \tilde{X} \in \mathbb{F} \) is trapezoidal (triangular, symmetric triangular, rectangle). Subject to the \( L_2 \)-distance, Grzegorzewski first proposed an interval approximation of a fuzzy number [9]. Afterwards, Ma et al. proposed the symmetric triangular approximation [15], Abbasbandy and Asady proposed the trapezoidal approximation preserving the core [1], and Grzegorzewski and Mrówka proposed the trapezoidal approximation preserving the expected interval [10–12]. Many properties were proved by Grzegorzewski and Mrówka [10]. Unfortunately, many scholars had pointed that some approximations may fail to be fuzzy numbers [4,5,18]. For filling the gap, many authors proposed their improved approximations [5,11,12,19,20]. While \( d(\cdot, \cdot) \) is not \( L_2 \)-distance, these type approximations were also investigated, see [2,6,7]. In addition, some other type approximations have been proposed, such as piece-wise parabolic approximations [13], polynomial approximations [3], and the nearest parametric approximations [16].

Let \( \lambda_L = \lambda_L(x) \) and \( \lambda_U = \lambda_U(x) \), \( x \in [0, 1] \), be nonnegative functions such that

\[
\int_0^1 \lambda_L(x) \, dx > 0 \quad \text{and} \quad \int_0^1 \lambda_U(x) \, dx > 0,
\]

which are called weighted functions on \([0, 1]\) in this paper. The weighted \( L_2 \)-distance \( d_J(\cdot, \cdot) \) on \( \mathbb{F} \) is defined as

\[
d_J(\tilde{A}, \tilde{B}) := \left[ \int_0^1 |A_L(x) - B_L(x)|^2 \lambda_L(x) \, dx + \int_0^1 |A_U(x) - B_U(x)|^2 \lambda_U(x) \, dx \right]^{1/2}.
\]

In particular, for the special case \( \lambda_L(x) = 1 - q \) and \( \lambda_U(x) = q \), where \( 0 < q < 1 \), it is easy to verify

\[
d_J(\tilde{A}, \tilde{B}) = \delta_{2,q}(\tilde{A}, \tilde{B}),
\]

where \( \delta_{2,q} \) is defined as (1.1). While \( \lambda_L = \lambda_U = \lambda(x) \) and

\[
\int_0^1 \lambda(x) \, dx = \frac{1}{2},
\]

Zeng and Li [21] proposed this type weighted triangular approximation. In [19], the author proved that Zeng and Li’s weighted triangular approximation of \([0, 1 - \sqrt{2}]\) is not a fuzzy number for any weighted function \( \lambda(x) \), see [19, Fact 2.2]. In this paper, we improve (or generalize) Zeng and Li’s triangular approximations under the weighted \( L_2 \)-distance \( d_J(\cdot, \cdot) \), defined as (1.3), and propose a more generalization by the name of weighted trapezoidal approximation which also generalizes the trapezoidal approximations under Grzegorzewski’s distance \( \delta_{2,q} \).

This paper is organized as follows. In Section 2, we introduce the space of all extended trapezoidal fuzzy numbers, which was first introduced in [20]. In fact, it is an inner product space. Some preliminaries are presented. In Section 3, weighted extended trapezoidal approximations of fuzzy numbers are discussed. Moreover, in Section 4 three triangular approximations are proposed. In Section 5, two main theorems are proved. In Section 6, we study algorithms for computing the weighted trapezoidal and triangular approximations. Some examples are illustrated. In Section 7, we present some relevant properties.

2. Preliminaries: inner product spaces and extended trapezoidal fuzzy numbers

An inner product on a (real) vector space \( \mathfrak{B} \) is a mapping \( \langle \cdot, \cdot \rangle : \mathfrak{B} \times \mathfrak{B} \to \mathbb{R} \) obeying the axioms:

1. \( \langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle \), for all \( u, v, w \in \mathfrak{B} \) and \( a, b \in \mathbb{R} \),
(2) \( \langle u, v \rangle = \langle v, u \rangle \), for all \( u, v \in \mathcal{H} \),
(3) \( \langle u, u \rangle \geq 0 \) for all \( u \in \mathcal{H} \), and \( \langle u, u \rangle = 0 \) if and only if \( u = 0 \).

A vector space equipped with an inner product is called an inner product space, and a completely inner product space is usually called a Hilbert space. Let \( \mathcal{H} \) be a closed convex subset of a Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\). Then, there uniquely exists an element in \( \mathcal{H} \) of smallest norm, see \([17, \text{Theorem 4.10, p. 79}]\). Let \( u \in \mathcal{H} \). Hence, there exists a unique element in \( \mathcal{H} \), denoted by \( P_\mathcal{H}(u) \), which minimizes the distance

\[
d(u, x) := \langle u - x, u - x \rangle^{1/2},
\]

where \( x \in \mathcal{H} \).

**Fact 2.1.** Let \( \mathcal{H} \) be a closed convex subset of a Hilbert space \( \mathcal{H} \), and \( u \in \mathcal{H} \). Then \( P_\mathcal{H}(u) \) equals the vector such that

\[
\langle u - P_\mathcal{H}(u), x - P_\mathcal{H}(u) \rangle \leq 0
\]

for all \( x \in \mathcal{H} \). Moreover, \( d(P_\mathcal{H}(u), P_\mathcal{H}(v)) \leq d(u, v) \) for all \( u, v \in \mathcal{H} \).

**Proof.** Let \( x \in \mathcal{H} \) and \( t \in (0, 1) \). We get \((1 - t)P_\mathcal{H}(u) + tx \in \mathcal{H} \), since \( \mathcal{H} \) is convex. Obviously,

\[
d^2(u, (1 - t)P_\mathcal{H}(u) + tx) = \langle u - P_\mathcal{H}(u) - t[x - P_\mathcal{H}(u)], u - P_\mathcal{H}(u) - t[x - P_\mathcal{H}(u)] \rangle \\
= d^2(u, P_\mathcal{H}(u)) - 2t \langle u - P_\mathcal{H}(u), x - P_\mathcal{H}(u) \rangle + t^2d^2(x, P_\mathcal{H}(u)).
\]

According to the definition of \( P_\mathcal{H}(u) \), we have

\[
d^2(u, (1 - t)P_\mathcal{H}(u) + tx) \geq d^2(u, P_\mathcal{H}(u))
\]

for all \( t \in (0, 1) \). That implies

\[
-2t \langle u - P_\mathcal{H}(u), x - P_\mathcal{H}(u) \rangle + t^2d^2(x, P_\mathcal{H}(u)) \geq 0,
\]

or equivalently,

\[
\langle u - P_\mathcal{H}(u), x - P_\mathcal{H}(u) \rangle \leq \frac{t}{2}d^2(x, P_\mathcal{H}(u)).
\]

Let \( t \to 0^+ \), we obtain

\[
\langle u - P_\mathcal{H}(u), x - P_\mathcal{H}(u) \rangle \leq 0
\]

for all \( x \in \mathcal{H} \). On the other hand, let \( v \) be an element in \( \mathcal{H} \) such that \( \langle u - v, y - v \rangle \leq 0 \) for all \( y \in \mathcal{H} \). Picking \( y = P_\mathcal{H}(u) \), we compute

\[
0 \geq \langle u - v, P_\mathcal{H}(u) - v \rangle \\
= \langle u - P_\mathcal{H}(u) + P_\mathcal{H}(u) - v, P_\mathcal{H}(u) - v \rangle \\
= -\langle u - P_\mathcal{H}(u), v - P_\mathcal{H}(u) \rangle + (P_\mathcal{H}(u) - v, P_\mathcal{H}(u) - v) \\
\geq (P_\mathcal{H}(u) - v, P_\mathcal{H}(u) - v).
\]

Hence, we obtain \( \langle P_\mathcal{H}(u) - v, P_\mathcal{H}(u) - v \rangle = 0 \). That implies \( P_\mathcal{H}(u) = v \).

Let \( u, v \in \mathcal{H} \) and set \( w = [u - P_\mathcal{H}(u)] - [v - P_\mathcal{H}(v)] \). We get

\[
\langle w, P_\mathcal{H}(u) - P_\mathcal{H}(v) \rangle = \langle u - P_\mathcal{H}(u), P_\mathcal{H}(v) - P_\mathcal{H}(u) \rangle - \langle v - P_\mathcal{H}(v), P_\mathcal{H}(u) - P_\mathcal{H}(v) \rangle \geq 0.
\]

Hence,

\[
\langle u - v, u - v \rangle = \langle w + P_\mathcal{H}(u) - P_\mathcal{H}(v), w + P_\mathcal{H}(u) - P_\mathcal{H}(v) \rangle \\
= \langle w, w \rangle + 2\langle w, P_\mathcal{H}(u) - P_\mathcal{H}(v) \rangle + \langle P_\mathcal{H}(u) - P_\mathcal{H}(v), P_\mathcal{H}(u) - P_\mathcal{H}(v) \rangle \\
\geq \langle P_\mathcal{H}(u) - P_\mathcal{H}(v), P_\mathcal{H}(u) - P_\mathcal{H}(v) \rangle.
\]

This completes the proof. \( \square \)
If \( A_L(x) \) and \( A_U(x) \) are both polynomials of degree less than or equal to 1, in this paper \( \hat{A} = [A_L(x), A_U(x)] \) is called an \textit{extended trapezoidal fuzzy number}, for detail refer to [20, Definition 2.1]. That is, \( \hat{A} \) is of the form \([l_1 + t_2 x, t_3 + t_4 x] \), \( t_1, t_2, t_3, t_4 \in \mathbb{R} \) (it may fail to satisfy \( t_1 \leq t_1 + t_2 \leq t_3 + t_4 \leq t_3 \)). The set of all extended trapezoidal fuzzy numbers is denoted by \( \mathcal{C} \). Recall that the fuzzy addition, fuzzy subtraction and fuzzy scalar multiplication on \( \mathbb{F} \) are defined as follows:

\[
\hat{A} + \hat{B} = [A_L(x) + B_L(x), A_U(x) + B_U(x)], \quad \hat{A} - \hat{B} = [A_L(x) - B_U(x), A_U(x) - B_L(x)], \quad r \cdot \hat{A} = \begin{cases} \lfloor r A_L(x), r A_U(x) \rfloor & \text{if } r \geq 0, \\ \lfloor r A_U(x), r A_L(x) \rfloor & \text{if } r < 0, \end{cases}
\]

where \( \hat{A} = [A_L(x), A_U(x)] \), \( \hat{B} = [B_L(x), B_U(x)] \) \( \in \mathbb{F} \) and \( r \in \mathbb{R} \). The inverse operation of fuzzy addition is not the fuzzy subtraction. In fact, it coincides with Hukuhara difference, denoted by \( \ominus \), see [14]. That means

\[
\hat{A} \ominus \hat{B} := [A_L(x) - B_L(x), A_U(x) - B_U(x)].
\]

Note that the Hukuhara difference between two trapezoidal fuzzy numbers may be not in \( \mathbb{F} \). For example, let \( \hat{A} = [-2 + 2x, 1 - x] \) and \( \hat{B} = [-1, 1 - 2x] \), then \( \hat{A} \ominus \hat{B} = [-1 + 2x, x] \not\in \mathbb{F} \). But, it is easy to verify that, \( \hat{A} \ominus \hat{B} \in \mathcal{C} \) for all \( \hat{A} \) and \( \hat{B} \in \mathcal{C} \). We also define another scalar multiplication on \( \mathcal{C} \), denoted by \( \odot \), as follows:

\[
r \odot \hat{A} := [r A_L(x), r A_U(x)],
\]

where \( \hat{A} \in \mathcal{C} \) and \( r \in \mathbb{R} \). Hence, \((\mathcal{C}, +, \odot)\) becomes a vector space over \( \mathbb{R} \). While \( \hat{A} \in \mathbb{F} \) and \( r \geq 0 \), we can easily verify

\[
r \odot \hat{A} = r \cdot \hat{A}.
\]

On the contrary, for \( r < 0 \) they are not equal.

Let \( \lambda_L = \hat{\lambda}_L(x) \) and \( \lambda_U = \hat{\lambda}_U(x) \) be two weighted functions on \([0,1]\). For arbitrary \( \hat{A} = [A_L(x), A_U(x)] \) and \( \hat{B} = [B_L(x), B_U(x)] \) \( \in \mathcal{C} \), the weighted \( L_2 \)-distance between \( \hat{A} \) and \( \hat{B} \) is similarly defined as (1.3). Unless otherwise stated, \( a, b, \omega_L, \omega_U, c, d \) are fixed to denote the following positive real numbers:

\[
a := \int_{0}^{1} \lambda_L(x) \, dx, \tag{2.1}
\]

\[
b := \int_{0}^{1} \lambda_U(x) \, dx, \tag{2.2}
\]

\[
\omega_L := a^{-1} \int_{0}^{1} x \lambda_L(x) \, dx, \tag{2.3}
\]

\[
\omega_U := b^{-1} \int_{0}^{1} x \lambda_U(x) \, dx, \tag{2.4}
\]

\[
c = \int_{0}^{1} (x - \omega_L)^2 \lambda_L(x) \, dx, \tag{2.5}
\]

\[
d = \int_{0}^{1} (x - \omega_U)^2 \lambda_U(x) \, dx. \tag{2.6}
\]

It is easy to verify that \( \omega_L < 1, \omega_U < 1 \),

\[
\int_{0}^{1} (x - \omega_L) \hat{\lambda}_L(x) \, dx = 0 \quad \text{and} \quad \int_{0}^{1} (x - \omega_U) \hat{\lambda}_U(x) \, dx = 0. \tag{2.7}
\]

In [19, Lemma 2.1], the author proved that

\[
\int_{0}^{1} g(x)(x - \omega) \hat{\lambda}(x) \, dx \leq 0
\]
for any decreasing function $g = g(x)$ and any weighted function $\lambda = \lambda(x)$ on $[0, 1]$, where
\[
\omega = \frac{\int_0^1 x\lambda(x)\,dx}{\int_0^1 \lambda(x)\,dx}.
\]
Hence, we have
\[
\int_0^1 g(x)(x - \omega_L)\lambda_L(x)\,dx \leq 0 \quad \text{and} \quad \int_0^1 g(x)(x - \omega_U)\lambda_U(x)\,dx \leq 0 \quad \text{(2.8)}
\]
for any decreasing function $g = g(x)$. In what follows, we always use the symbol $(l, u, x, y)_\lambda$ to rewrite an extended trapezoidal fuzzy number with $x$-cuts
\[
[l + x(x - \omega_L), u - y(x - \omega_U)],
\]
where $l, u, x, y \in \mathbb{R}$. Proposition 2.2 provides a formula for computing the weighted $L_2$-distance $d_\lambda(\tilde{A}, \tilde{B})$ in terms of their new representations.

**Proposition 2.2.** Let $\tilde{A} = (l_1, u_1, x_1, y_1)_\lambda$ and $\tilde{B} = (l_2, u_2, x_2, y_2)_\lambda$ be two extended trapezoidal fuzzy numbers. Then,
\[
d^2_\lambda(\tilde{A}, \tilde{B}) = a(l_1 - l_2)^2 + b(u_1 - u_2)^2 + c(x_1 - x_2)^2 + d(y_1 - y_2)^2.
\]

**Proof.** It is clear that
\[
\tilde{A} = [l_1 + x_1(x - \omega_L), u_1 - y_1(x - \omega_U)] \quad \text{and} \quad \tilde{B} = [l_2 + x_2(x - \omega_L), u_2 - y_2(x - \omega_U)].
\]
From (1.3), we compute
\[
d^2_\lambda(\tilde{A}, \tilde{B}) = \int_0^1 |l_1 + x_1(x - \omega_L) - l_2 - x_2(x - \omega_L)|^2\lambda_L(x)\,dx
\]
\[
+ \int_0^1 |u_1 - y_1(x - \omega_U) - u_2 + y_2(x - \omega_U)|^2\lambda_U(x)\,dx \quad \text{(by (2.7))}
\]
\[
= \int_0^1 [(l_1 - l_2)^2 + (x_1 - x_2)^2(x - \omega_L)^2]\lambda_L(x)\,dx
\]
\[
+ \int_0^1 [(u_1 - u_2)^2 + (y_1 - y_2)^2(x - \omega_U)^2]\lambda_U(x)\,dx.
\]
Consequently, by applying (2.1), (2.2), (2.5), and (2.6), we completes the proof. \qed

**Proposition 2.3.** Let $\tilde{A} = (l, u, x, y)_\lambda$ be an extended trapezoidal fuzzy number. Then,

1. $\tilde{A}$ is trapezoidal iff $x, y \geq 0$ and $l - u + (1 - \omega_L)x + (1 - \omega_U)y \leq 0$,
2. $\tilde{A}$ is triangular iff $x, y \geq 0$ and $l - u + (1 - \omega_L)x + (1 - \omega_U)y = 0$.

**Proof.** It is clear that
\[
\tilde{A} = [l + x(x - \omega_L), u - y(x - \omega_U)] = [l - \omega_Lx + xz, u + \omega_Uy - yz].
\]
From constraint (1.2), we find that $\tilde{A}$ is trapezoidal iff
\[
l - \omega_Lx \leq l + (1 - \omega_L)x \leq u - (1 - \omega_U)y \leq u + \omega_Uy.
\]
Hence, the above inequalities are equivalent to $x, y \geq 0$ and
\[
(1 - \omega_L)x + (1 - \omega_U)y \leq u - l.
\]
In the same vein, we can prove the other one. \qed
Let \( \tilde{\mathcal{A}} = (l_1, u_1, x_1, y_1)_\tilde{\lambda}, \) \( \tilde{\mathcal{B}} = (l_2, u_2, x_2, y_2)_\tilde{\lambda} \in \mathcal{E}, \) and \( r \in \mathbb{R}. \) With respect to the new representation, we have
\[
\tilde{\mathcal{A}} + \tilde{\mathcal{B}} = (l_1 + l_2, u_1 + u_2, x_1 + x_2, y_1 + y_2)_\tilde{\lambda},
\]
\[
\tilde{\mathcal{A}} \odot \tilde{\mathcal{B}} := (l_1 - l_2, u_1 - u_2, x_1 - x_2, y_1 - y_2)_\tilde{\lambda},
\]
\[
r \circ \tilde{\mathcal{A}} = (rl_1, ru_1, rx_1, ry_1)_\tilde{\lambda}
\]
for all \( \tilde{\mathcal{A}} \) and \( \tilde{\mathcal{B}} \in \mathcal{E} \) and \( r \in \mathbb{R}. \) In the previous, we claim that \((\mathcal{E}, +, \odot)\) is a vector space. Now, we define an inner product on \( \mathcal{E} \) as
\[
\langle \tilde{\mathcal{A}}, \tilde{\mathcal{B}} \rangle := al_1l_2 + bu_1u_2 + cx_1x_2 + dy_1y_2,
\]
where \( a, b, c, d \) are positive real numbers defined by (2.1), (2.2), (2.5), and (2.6). Since every finite dimensional inner product space is complete, \((\mathcal{E}, \langle \cdot, \cdot \rangle)\) is a Hilbert space. By applying Proposition 2.2, it is easy to get the following proposition.

**Proposition 2.4.** Let \( \tilde{\mathcal{A}} = (l_1, u_1, x_1, y_1)_\tilde{\lambda} \) and \( \tilde{\mathcal{B}} = (l_2, u_2, x_2, y_2)_\tilde{\lambda} \) be two extended trapezoidal fuzzy numbers. Then,
\[
d^2(\tilde{\mathcal{A}}, \tilde{\mathcal{B}}) = \langle \tilde{\mathcal{A}} \odot \tilde{\mathcal{B}}, \tilde{\mathcal{A}} \odot \tilde{\mathcal{B}} \rangle.
\]

### 3. Weighted extended trapezoidal approximations

Let \( \tilde{\mathcal{A}} = [A_L(x), A_U(x)] \) be an arbitrary fuzzy number. The **weighted extended trapezoidal approximation** of \( \tilde{\mathcal{A}} \) is the element in \( \mathcal{E} \) that minimizes the weighted \( L_2 \)-distance \( d_\lambda(\tilde{\mathcal{A}}, \tilde{\mathcal{X}}) \), where \( \tilde{\mathcal{X}} \in \mathcal{E}. \) In this paper, we denote by
\[
T_e(\tilde{\mathcal{A}}) = (l_e, u_e, x_e, y_e)_\tilde{\lambda}
\]
the extended trapezoidal fuzzy number determined by the following equalities:
\[
l_e = l_e(\tilde{\mathcal{A}}) := a^{-1} \int_0^1 A_L(x) \tilde{\lambda}_L(x) \, dx,
\]
\[
u_e = u_e(\tilde{\mathcal{A}}) := b^{-1} \int_0^1 A_U(x) \tilde{\lambda}_U(x) \, dx,
\]
\[
x_e = x_e(\tilde{\mathcal{A}}) := c^{-1} \int_0^1 A_L(x)(x - \omega_L) \tilde{\lambda}_L(x) \, dx,
\]
\[
y_e = y_e(\tilde{\mathcal{A}}) := -d^{-1} \int_0^1 A_U(x)(x - \omega_U) \tilde{\lambda}_U(x) \, dx.
\]

From the definition of fuzzy numbers, we find
\[
A_L(x) \leq A_L(1) \leq A_U(1) \leq A_U(x)
\]
for all \( x \in [0, 1] \). By applying (2.1) and (2.2), we have
\[
l_e = a^{-1} \int_0^1 A_L(x) \tilde{\lambda}_L(x) \, dx \leq a^{-1} \int_0^1 A_L(1) \tilde{\lambda}_L(x) \, dx = A_L(1)
\]
and
\[
u_e = b^{-1} \int_0^1 A_U(x) \tilde{\lambda}_U(x) \, dx \geq b^{-1} \int_0^1 A_U(1) \tilde{\lambda}_U(x) \, dx = A_U(1),
\]
so that
\[
u_e - l_e \geq 0.
\]

In the following, we will show that \( T_e(\tilde{\mathcal{A}}) \) is equal to the weighted extended trapezoidal approximation of \( \tilde{\mathcal{A}}. \)
Lemma 3.1. Let $\tilde{A} = [A_L(x), A_U(x)]$ be a fuzzy number, and $T_c(\tilde{A}) = (l_c, u_c, x_c, y_c)$ be defined as (3.1)–(3.4). Then the following equalities hold for all $r$.

\[
\int_{0}^{1} [A_L(x) - l_c - x_c(x - \omega_L)](r + sx)\lambda_L(x) \, dx = 0 \tag{3.6}
\]

and

\[
\int_{0}^{1} [A_U(x) - u_c + y_c(x - \omega_U)](r + sx)\lambda_U(x) \, dx = 0 \tag{3.7}
\]

hold for all $r, s \in \mathbb{R}$.

Proof. From (3.1), we find

\[
\int_{0}^{1} A_L(x)\lambda_L(x) \, dx = al_c.
\]

By applying (2.1) and (2.7), we get

\[
al_c = \int_{0}^{1} l_c\lambda_L(x) \, dx = \int_{0}^{1} [l_c + x_c(x - \omega_L)]\lambda_L(x) \, dx.
\]

We conclude

\[
\int_{0}^{1} [A_L(x) - l_c - x_c(x - \omega_L)]\lambda_L(x) \, dx = 0.
\]

To obtain (3.6), it suffices to prove

\[
\int_{0}^{1} [A_L(x) - l_c - x_c(x - \omega_L)](x - \omega_L)\lambda_L(x) \, dx = 0. \tag{3.8}
\]

From (3.3), we find

\[
\int_{0}^{1} A_L(x)(x - \omega_L)\lambda_L(x) \, dx = cx_c.
\]

By applying (2.5) and (2.7), we get

\[
cx_c = \int_{0}^{1} x_c(x - \omega_L)^2\lambda_L(x) \, dx = \int_{0}^{1} [l_c + x_c(x - \omega_L)](x - \omega_L)\lambda_L(x) \, dx.
\]

This proves (3.8), and so does (3.6). In the same vein, we can prove (3.7). \qed

Proposition 3.2. Let $\tilde{A}$ be a fuzzy number, and $T_c(\tilde{A}) = (l_c, u_c, x_c, y_c)$ be defined as (3.1)–(3.4). Then,

\[
d^2_f(\tilde{A}, \tilde{X}) = d_f^2(\tilde{A}, T_c(\tilde{A})) + d^2_f(T_c(\tilde{A}), \tilde{X})
\]

for every extended trapezoidal fuzzy number $\tilde{X}$.

Proof. Let $\tilde{A} = [A_L(x), A_U(x)]$ and $T_c(\tilde{A}) = [T_c(\tilde{A})_L(x), T_c(\tilde{A})_U(x)]$. It is clear that

\[
T_c(\tilde{A})_L(x) = l_c + x_c(x - \omega_L) \quad \text{and} \quad T_c(\tilde{A})_U(x) = u_c - y_c(x - \omega_U).
\]

Let $\tilde{X} = [t_1 + t_2x, t_3 + t_4x] \in \mathbb{C}$. Applying Lemma 3.1 we obtain

\[
\int_{0}^{1} [A_L(x) - T_c(\tilde{A})_L(x)](T_c(\tilde{A})_L(x) - (t_1 + t_2x)]\lambda_L(x) \, dx = 0.
\]
and
\[
\int_0^1 [A_L(z) - T_c(\tilde{A})_L(z)][T_c(\tilde{A})_U(z) - (t_3 + t_4z)]\lambda_U(z) \, dz = 0.
\]

Keeping the above two equalities in hand, we compute from (1.3):
\[
d_\lambda^2(\tilde{A}, \tilde{X}) = \int_0^1 [A_L(z) - (t_1 + t_2z)]^2\lambda_L(z) \, dz + \int_0^1 [A_U(z) - (t_3 + t_4z)]^2\lambda_U(z) \, dz
\]
\[
= \int_0^1 [A_L(z) - T_c(\tilde{A})_L(z) + T_c(\tilde{A})_U(z) - (t_1 + t_2z)]^2\lambda_L(z) \, dz
\]
\[
+ \int_0^1 [A_U(z) - T_c(\tilde{A})_U(z) + T_c(\tilde{A})_L(z) - (t_3 + t_4z)]^2\lambda_U(z) \, dz
\]
\[
= \int_0^1 [A_L(z) - T_c(\tilde{A})_L(z)]^2\lambda_L(z) \, dz + \int_0^1 [T_c(\tilde{A})_L(z) - (t_1 + t_2z)]^2\lambda_L(z) \, dz
\]
\[
+ \int_0^1 [A_U(z) - T_c(\tilde{A})_U(z)]^2\lambda_U(z) \, dz + \int_0^1 [T_c(\tilde{A})_U(z) - (t_3 + t_4z)]^2\lambda_U(z) \, dz
\]
\[
= d_\lambda^2(\tilde{A}, T_c(\tilde{A}))^2 + d_\lambda^2(T_c(\tilde{A}), \tilde{X})^2.
\]

This completes the proof. \(\square\)

**Proposition 3.3.** Let \(\tilde{A}\) be a fuzzy number. Then \(T_c(\tilde{A}) = (l_c, u_c, x_c, y_c)_\lambda\) equals the weighted extended trapezoidal approximation of \(\tilde{A}\).

**Proof.** Obviously, Proposition 3.2 implies
\[
d_\lambda(\tilde{A}, T_c(\tilde{A})) \leq d_\lambda(\tilde{A}, \tilde{X})
\]
for all \(\tilde{X} \in \mathcal{E}\). Hence, \(T_c(\tilde{A})\) is the weighted extended trapezoidal approximation of \(\tilde{A}\). \(\square\)

While \(\lambda_L(z) = \lambda_U(z) = 1\), from (2.1)–(2.6) it is easy to verify that
\[
a = b = 1, \quad \omega_L = \omega_U = \frac{1}{2} \quad \text{and} \quad c = d = \frac{1}{2}.
\]
Substituting into (3.1)–(3.4), we will prove that (3.1)–(3.4) coincide with [19, Eqs. (3.6)–(3.9)], respectively. This shows that \(T_c\) equals Grzegorzewski and Mrówka’s trapezoidal approximation [10] in the special case.

**Lemma 3.4.** Let \(\tilde{A}\) be a fuzzy number, and \(T_c(\tilde{A}) = (l_c, u_c, x_c, y_c)_\lambda\) be its weighted extended trapezoidal approximation. Then, \(x_c, y_c \geq 0\).

**Proof.** Let \(\tilde{A} = (A_L(z), A_U(z)) \in \mathbb{F}\). From definition of fuzzy numbers, we find that \(A_L\) is increasing and \(A_U\) is decreasing. By applying (2.8), we obtain
\[
\int_0^1 A_L(z)(x - \omega_L)\lambda_L(z) \, dz \geq 0 \quad \text{and} \quad \int_0^1 A_U(z)(x - \omega_U)\lambda_U(z) \, dz \geq 0.
\]
Since \(c, d > 0\), applying (3.3) and (3.4) we prove \(x_c, y_c \geq 0\). \(\square\)

4. Three extended triangular approximations

In Section 2, we have shown that \((\mathcal{E}, \langle \cdot, \cdot \rangle_\lambda)\) is a Hilbert space, where \(\langle \cdot, \cdot \rangle_\lambda\) is defined as (2.11). In this paper, an element in \(\mathcal{E}\) is also called a vector. It is easy to show that the subset
\[
\mathcal{J} := \{(l, u, x, y)_\lambda \in \mathcal{E} \mid l - u + (1 - \omega_L)x + (1 - \omega_U)y = 0\}
\]
is a subspace of \(\mathcal{E}\). Every element in \(\mathcal{J}\) is called extended triangular fuzzy number. From Proposition 2.3(2), we can find that each triangular fuzzy number is in \(\mathcal{J}\). Let \(\tilde{A} \in \mathcal{E}\) and \(T_c(\tilde{A})\) be its weighted extended trapezoidal approximation. We
denote by $Z(\tilde{A})$ the projection of $T_e(\tilde{A})$ onto the subspace $\mathcal{Z}$ under the inner product $\langle \cdot, \cdot \rangle_L$. In fact, while $\lambda_L = \lambda_U = \lambda(x)$ and
\[
\int_0^1 \lambda(x) dx = \frac{1}{2},
\]
$Z(\tilde{A})$ coincides with Zeng and Li’s [21] weighted triangular approximation of $\tilde{A}$. Hence, we use the capital letter “$Z$” to denote this approximation. Notice that, by applying (2.11), we have
\[
\lambda_L = \lambda_U = \lambda(x) = \lambda_L = \lambda_U = \lambda(x) + \lambda_L, \quad (a^{-1}, -b^{-1}, c^{-1}(1 - \omega_L), d^{-1}(1 - \omega_U))_\lambda.
\]
Hence, the normal vector of $\mathcal{Z}$ is
\[
\tilde{N} := (a^{-1}, -b^{-1}, c^{-1}(1 - \omega_L), d^{-1}(1 - \omega_U))_\lambda.
\]
That means $\tilde{X} \in \mathcal{Z}$ iff $\langle \tilde{N}, \tilde{X} \rangle_\lambda = 0$. (4.2)
Furthermore, we can obtain (see Fig. 1)
\[
Z(\tilde{A}) = T_e(\tilde{A}) \Theta \frac{\langle T_e(\tilde{A}), \tilde{N} \rangle_\lambda}{\langle \tilde{N}, \tilde{N} \rangle_\lambda} \ominus \tilde{N},
\]
where $\Theta$ and $\ominus$ are defined as (2.9) and (2.10), respectively.

Let $\tilde{A}$ be an arbitrary fuzzy number. The weighted extended triangular approximation of $\tilde{A}$ is the element in $\mathcal{Z}$ that minimizes the weighted $L_2$-distance $d_\lambda(\tilde{A}, \tilde{X})$, where $\tilde{X} \in \mathcal{Z}$. Since $Z(\tilde{A})$ is the projection of $T_e(\tilde{A})$ onto the subspace $\mathcal{Z}$, we get
\[
d_\lambda(T_e(\tilde{A}), \tilde{X}) \geq d_\lambda(T_e(\tilde{A}), Z(\tilde{A}))
\]
for all $\tilde{X} \in \mathcal{Z}$. By applying Proposition 3.2, we compute
\[
d_\lambda^2(\tilde{A}, \tilde{X}) = d_\lambda^2(T_e(\tilde{A}), \tilde{X}) = d_\lambda^2(T_e(\tilde{A}), Z(\tilde{A}))
\]
Hence, obtain the following theorem.

**Theorem 4.1.** Let $\tilde{A}$ be an arbitrary fuzzy number. Then, $Z(\tilde{A})$ equals the weighted extended triangular approximation of $\tilde{A}$.

Let us define another two subspaces of $\mathcal{E}$, which are both contained in $\mathcal{Z}$, as follows:
\[
\mathcal{Z}_x := \{(l, u, x, 0)_L \in \mathcal{E} | l - u + (1 - \omega_L)x = 0\}
\]
Proof. Since \( \tilde{E}_3 := (0, 0, 1, 0) \), \( \tilde{E}_4 := (0, 0, 0, 1) \) \( \in \mathfrak{C} \), it is easy to prove the following lemma.

**Lemma 4.2.** For a vector \( \tilde{X} \in \mathfrak{C} \), we have

1. \( \tilde{X} \in \mathfrak{Z}_x \) iff \( \langle \tilde{X}, \tilde{N} \rangle \rangle = \langle \tilde{X}, \tilde{E}_4 \rangle \rangle = 0 \) and
2. \( \tilde{X} \in \mathfrak{Z}_y \) iff \( \langle \tilde{X}, \tilde{N} \rangle \rangle = \langle \tilde{X}, \tilde{E}_3 \rangle \rangle = 0 \).

Consequently, for arbitrary fuzzy number \( \tilde{A} \) let us define another two vectors \( \tilde{N}_x = \tilde{N}_x(\tilde{A}) \) and \( \tilde{N}_y = \tilde{N}_y(\tilde{A}) \) in \( \mathfrak{C} \) as follows:

\[
\tilde{N}_x := \frac{(T_e(\tilde{A}), \tilde{N}) \rangle - y_e(1 - \omega_U)}{(\tilde{N}, \tilde{N}) \rangle - d^{-1}(1 - \omega_U)^2} \odot \tilde{N} + \frac{y_e(\tilde{N}, \tilde{N}) \rangle - d^{-1}(1 - \omega_U)(T_e(\tilde{A}), \tilde{N}) \rangle}{(\tilde{N}, \tilde{N}) \rangle - d^{-1}(1 - \omega_U)^2} \odot \tilde{E}_4
\]

and

\[
\tilde{N}_y := \frac{(T_e(\tilde{A}), \tilde{N}) \rangle - x_e(1 - \omega_L)}{(\tilde{N}, \tilde{N}) \rangle - c^{-1}(1 - \omega_L)^2} \odot \tilde{N} + \frac{x_e(\tilde{N}, \tilde{N}) \rangle - c^{-1}(1 - \omega_L)(T_e(\tilde{A}), \tilde{N}) \rangle}{(\tilde{N}, \tilde{N}) \rangle - c^{-1}(1 - \omega_L)^2} \odot \tilde{E}_3,
\]

where \( T_e(\tilde{A}) = (l_e, u_e, x_e, y_e) \) is the weighted extended trapezoidal approximation of \( \tilde{A} \). By Lemma 4.2, we have

\[
\langle \tilde{X}, \tilde{N}_x \rangle \rangle = 0 \text{ for all } \tilde{X} \in \mathfrak{Z}_x
\]

and

\[
\langle \tilde{Y}, \tilde{N}_y \rangle \rangle = 0 \text{ for all } \tilde{Y} \in \mathfrak{Z}_y.
\]

Similarly, let \( Z_x(\tilde{A}) \) and \( Z_y(\tilde{A}) \) denote the projections of \( T_e(\tilde{A}) \) onto the subspaces \( \mathfrak{Z}_x \) and \( \mathfrak{Z}_y \), respectively.

**Lemma 4.3.** Let \( \tilde{A} = [A_L(\alpha), A_U(\alpha)] \) be an arbitrary fuzzy number. Then

\[
Z_x(\tilde{A}) = T_e(\tilde{A}) \odot \tilde{N}_x \quad \text{and} \quad Z_y(\tilde{A}) = T_e(\tilde{A}) \odot \tilde{N}_y,
\]

where \( \tilde{N}_x \) and \( \tilde{N}_y \) are computed by (4.4) and (4.5).

**Proof.** Since \( \tilde{N}_x \) is perpendicular to the subspace \( \mathfrak{Z}_x \) (by (4.6)), to obtain \( Z_x(\tilde{A}) = T_e(\tilde{A}) \odot \tilde{N}_x \) it suffices to prove that

\[
T_e(\tilde{A}) \odot \tilde{N}_x \in \mathfrak{Z}_x,
\]

see Fig. 1. Obviously, from (2.11) we find

\[
\langle \tilde{E}_4, \tilde{N} \rangle \rangle = \langle \tilde{N}, \tilde{E}_4 \rangle \rangle = 1 - \omega_U, \quad \langle \tilde{E}_4, \tilde{E}_4 \rangle \rangle = d \quad \text{and} \quad \langle T_e(\tilde{A}), \tilde{E}_4 \rangle \rangle = dy_e.
\]

Keeping (4.4) in hand, we compute

\[
\langle \tilde{N}_x, \tilde{N} \rangle \rangle = \frac{(T_e(\tilde{A}), \tilde{N}) \rangle - y_e(1 - \omega_U)}{(\tilde{N}, \tilde{N}) \rangle - d^{-1}(1 - \omega_U)^2} \langle \tilde{N}, \tilde{N} \rangle \rangle + \frac{y_e(\tilde{N}, \tilde{N}) \rangle - d^{-1}(1 - \omega_U)(T_e(\tilde{A}), \tilde{N}) \rangle}{(\tilde{N}, \tilde{N}) \rangle - d^{-1}(1 - \omega_U)^2}
\]

and

\[
\langle \tilde{N}_x, \tilde{E}_4 \rangle \rangle = \frac{y_e(1 - \omega_U)}{(\tilde{N}, \tilde{N}) \rangle - d^{-1}(1 - \omega_U)^2}(1 - \omega_U) + \frac{y_e(\tilde{N}, \tilde{N}) \rangle - d^{-1}(1 - \omega_U)(T_e(\tilde{A}), \tilde{N}) \rangle d}{(\tilde{N}, \tilde{N}) \rangle - d^{-1}(1 - \omega_U)^2}
\]

and

\[
\langle \tilde{N}_y, \tilde{N} \rangle \rangle = \frac{(T_e(\tilde{A}), \tilde{N}) \rangle - x_e(1 - \omega_L)}{(\tilde{N}, \tilde{N}) \rangle - c^{-1}(1 - \omega_L)^2} \odot \tilde{N} + \frac{x_e(\tilde{N}, \tilde{N}) \rangle - c^{-1}(1 - \omega_L)(T_e(\tilde{A}), \tilde{N}) \rangle}{(\tilde{N}, \tilde{N}) \rangle - c^{-1}(1 - \omega_L)^2} \odot \tilde{E}_3,
\]

and

\[
\langle \tilde{N}_y, \tilde{E}_3 \rangle \rangle = \frac{c^{-1}(1 - \omega_L)}{(\tilde{N}, \tilde{N}) \rangle - c^{-1}(1 - \omega_L)^2}(1 - \omega_L) + \frac{x_e(\tilde{N}, \tilde{N}) \rangle - c^{-1}(1 - \omega_L)(T_e(\tilde{A}), \tilde{N}) \rangle d}{(\tilde{N}, \tilde{N}) \rangle - c^{-1}(1 - \omega_L)^2}
\]
We hence have

\[ \langle T_e(\tilde{A}) \bigcap N_x, \tilde{N} \rangle_{\lambda} = \langle T_e(\tilde{A}), \tilde{N} \rangle_{\lambda} - \langle N_x, \tilde{N} \rangle_{\lambda} = 0 \]

and

\[ \langle T_e(\tilde{A}) \bigcap N_x, \tilde{E}_4 \rangle_{\lambda} = \langle T_e(\tilde{A}), \tilde{E}_4 \rangle_{\lambda} - \langle N_x, \tilde{E}_4 \rangle_{\lambda} = 0. \]

By applying Lemma 4.2(1), we obtain \( T_e(\tilde{A}) \bigcap N_x \in \tilde{3}_x \).

In the same way, by applying (4.5), (4.7) and Lemma 4.2(2) we can prove the other one. □

5. Weighted trapezoidal and triangular approximations

Let \( \tilde{A} \in \mathbb{F} \), and let \( T(\tilde{A}) \) denote its weighted trapezoidal approximation. That means \( T(\tilde{A}) \) is the trapezoidal fuzzy number that minimizes the weighted \( L_2 \)-distance \( d_L(\tilde{A}, \tilde{X}) \), where \( \tilde{X} \in \mathbb{F} \) is trapezoidal. For computing \( T(\tilde{A}) \), let us define four subsets of \( \mathbb{F} \) as follows:

\[ \Gamma_1 := \{ A \in \mathbb{F} \mid \langle T_e(A), \tilde{N} \rangle_{\lambda} \leq 0 \}, \]

\[ \Gamma_2 := \{ A \in \mathbb{F} \mid 0 < \langle T_e(A), \tilde{N} \rangle_{\lambda} \leq \langle N, \tilde{N} \rangle_{\lambda} \cdot \min \{ c x_e(1 - \omega_L)^{-1}, d y_e(1 - \omega_U)^{-1} \} \}, \]

\[ \Gamma_3 := \{ A \in \mathbb{F} \mid \langle T_e(A), \tilde{N} \rangle_{\lambda} > \langle N, \tilde{N} \rangle_{\lambda} \cdot d y_e(1 - \omega_U)^{-1} \}, \]

\[ \Gamma_4 := \{ A \in \mathbb{F} \mid \langle T_e(A), \tilde{N} \rangle_{\lambda} > \langle N, \tilde{N} \rangle_{\lambda} \cdot c x_e(1 - \omega_L)^{-1} \}. \]

Lemma 5.1. The four subsets \( \Gamma_i, 1 \leq i \leq 4 \), form a partition of \( \mathbb{F} \).

Proof. It is clear that

\[ \bigcup_{i=1}^{4} \Gamma_i = \mathbb{F} \quad \text{and} \quad \Gamma_2 \cap \Gamma_1 = \Gamma_2 \cap \Gamma_3 = \Gamma_2 \cap \Gamma_4 = \emptyset. \]

In Lemma 3.4, we have proved \( x_e, y_e \geq 0 \), so that

\[ \langle \tilde{N}, \tilde{N} \rangle_{\lambda} \cdot d y_e(1 - \omega_U)^{-1} \geq 0 \quad \text{and} \quad \langle \tilde{N}, \tilde{N} \rangle_{\lambda} \cdot c x_e(1 - \omega_L)^{-1} \geq 0. \]

That means

\[ \tilde{A} \in \Gamma_3 \cup \Gamma_4 \quad \text{implies} \quad \langle T_e(\tilde{A}), \tilde{N} \rangle_{\lambda} > 0. \] (5.1)

Hence, \( \Gamma_1 \cap \Gamma_3 = \Gamma_1 \cap \Gamma_4 = \emptyset \). Suppose on a contrary that there exists a fuzzy number \( \tilde{A} \in \Gamma_3 \cap \Gamma_4 \). That means

\[ \langle \tilde{N}, \tilde{N} \rangle_{\lambda} \cdot (1 - \omega_U) y_e < \langle T_e(\tilde{A}), \tilde{N} \rangle_{\lambda} \cdot d^{-1}(1 - \omega_U)^2 \]

and

\[ \langle \tilde{N}, \tilde{N} \rangle_{\lambda} \cdot (1 - \omega_L) x_e < \langle T_e(\tilde{A}), \tilde{N} \rangle_{\lambda} \cdot c^{-1}(1 - \omega_L)^2, \]

so that

\[ \langle \tilde{N}, \tilde{N} \rangle_{\lambda} [(1 - \omega_L) x_e + (1 - \omega_U) y_e] < \langle T_e(\tilde{A}), \tilde{N} \rangle_{\lambda} [c^{-1}(1 - \omega_L)^2 + d^{-1}(1 - \omega_U)^2]. \] (5.2)

Notice that

\[ (1 - \omega_L) x_e + (1 - \omega_U) y_e = \langle T_e(\tilde{A}), \tilde{N} \rangle_{\lambda} + (u_e - l_e) \] (5.3)

and

\[ \langle \tilde{N}, \tilde{N} \rangle_{\lambda} = a^{-1} + b^{-1} + c^{-1}(1 - \omega_L)^2 + d^{-1}(1 - \omega_U)^2. \] (5.4)
Substituting the above two equalities into (5.2), we can obtain

\[
\langle \tilde{N}, \tilde{N} \rangle \cdot (u_e - l_e) < \langle T_c(\tilde{A}), \tilde{N} \rangle \cdot \left[ c^{-1}(1 - \omega_L)^2 + d^{-1}(1 - \omega_U)^2 - \langle \tilde{N}, \tilde{N} \rangle \right]
\]

\[
= -\langle T_c(\tilde{A}), \tilde{N} \rangle \cdot (a^{-1} + b^{-1}).
\]

Since \(u_e - l_e \geq 0\) (from (3.5)) and \(\langle \tilde{N}, \tilde{N} \rangle \geq 0\), we obtain \(\langle T_c(\tilde{A}), \tilde{N} \rangle < 0\), which contradicts (5.1). Thus, \(\Gamma_3 \cap \Gamma_4 = \emptyset\).

**Theorem 5.2.** Let \(\tilde{A}\) be a fuzzy number, \(T_c(\tilde{A}) = (l_e, u_e, x_e, y_e)\) be its weighted extended trapezoidal approximation, and let \(Z(\tilde{A}), Z_x(\tilde{A}),\) and \(Z_y(\tilde{A})\) be the projections of \(T_c(\tilde{A})\) onto the subspaces \(3, 3_x,\) and \(3_y,\) respectively. Then the weighted trapezoidal approximation \(T(\tilde{A})\) can be calculated in the following cases:

1. \(\tilde{A} \in \Gamma_1\) implies \(T(\tilde{A}) = T_c(\tilde{A})\),
2. \(\tilde{A} \in \Gamma_2\) implies \(T(\tilde{A}) = Z(\tilde{A})\),
3. \(\tilde{A} \in \Gamma_3\) implies \(T(\tilde{A}) = Z_x(\tilde{A})\),
4. \(\tilde{A} \in \Gamma_4\) implies \(T(\tilde{A}) = Z_y(\tilde{A})\).

**Proof.** Let \(\tilde{X}\) be an arbitrary trapezoidal fuzzy number. Applying Proposition 3.2, we have

\[
d^2(\tilde{A}, \tilde{X}) = d^2(T_c(\tilde{A}), \tilde{X}) + d^2(T_c(\tilde{A}), \tilde{X}).
\]

To minimize \(d(\tilde{A}, \tilde{X})\), it suffices to minimize \(d^2(T_c(\tilde{A}), \tilde{X})\) since \(d^2(\tilde{A}, T_c(\tilde{A}))\) is constant. Consequently, applying Proposition 2.4 we get

\[
d^2(T_c(\tilde{A}), \tilde{X}) = (T_c(\tilde{A}) \ominus \tilde{X}, T_c(\tilde{A}) \ominus \tilde{X})_\lambda,
\]

where \("\ominus\)" is defined as (2.9). From Proposition 2.3(1), we find the set of all trapezoidal fuzzy numbers is equal to

\[
\{ (l, u, x, y) \in \mathcal{E} | x, y \geq 0, l - u + (1 - \omega_L)x + (1 - \omega_U)y \leq 0 \},
\]

hence it is closed and convex. By applying Fact 2.1 and setting \(u = T_c(\tilde{A})\), we can find that \(T(\tilde{A})\) equals the trapezoidal fuzzy number such that

\[
(T_c(\tilde{A}) \ominus T(\tilde{A}), \tilde{X} \ominus T(\tilde{A}))_\lambda \leq 0
\]

for all trapezoidal fuzzy numbers \(\tilde{X}\).

**Case (1):** According to Lemma 3.4, we have \(x_e, y_e \geq 0\). By applying (5.3), \(\tilde{A} \in \Gamma_1\) implies

\[
l_e - u_e + (1 - \omega_L)x_e + (1 - \omega_U)y_e = (T_c(\tilde{A}), \tilde{N})_\lambda \leq 0.
\]

Consequently applying Proposition 2.3(1), we get \(T_c(\tilde{A})\) is trapezoidal. Hence, (5.5) holds by setting \(T(\tilde{A}) = T_c(\tilde{A})\).

**Case (2):** First, we claim that if

\[
(T_c(\tilde{A}), \tilde{N})_\lambda \leq (\tilde{N}, \tilde{N})_\lambda \cdot \min\{c x_e(1 - \omega_L)^{-1}, d y_e(1 - \omega_U)^{-1}\}
\]

then \(Z(\tilde{A})\) is triangular. Let \(Z(\tilde{A}) = (l_z, u_z, x_z, y_z)_\lambda\). From the definitions of \(3\), by applying Proposition 2.3(2) it suffices to prove \(x_z, y_z \geq 0\). From (4.3), we find

\[
x_z = x_e - \frac{(T_c(\tilde{A}), \tilde{N})_\lambda}{(\tilde{N}, \tilde{N})_\lambda} \cdot c^{-1}(1 - \omega_L)
\]

\[
= \frac{c^{-1}(1 - \omega_L)}{(\tilde{N}, \tilde{N})_\lambda} \cdot \frac{c x_e(1 - \omega_L)^{-1} - (T_c(\tilde{A}), \tilde{N})_\lambda}{(\tilde{N}, \tilde{N})_\lambda}
\]

(5.6)
and

\[ y_z = y_c - \frac{(T_c(\tilde{A}), \tilde{N})_\lambda}{(\tilde{N}, \tilde{N})_\lambda} \cdot d^{-1} \]

\[ = \frac{d^{-1}(1 - \omega_U)}{(\tilde{N}, \tilde{N})_\lambda} \cdot d y_c (1 - \omega_U)^{-1} - \frac{(T_c(\tilde{A}), \tilde{N})_\lambda}{(\tilde{N}, \tilde{N})_\lambda}. \]  

(5.7)

Hence, \( x_z, y_z \geq 0 \).

By applying Fact 2.1, it suffices to prove (5.5) by setting \( T(\tilde{A}) = Z(\tilde{A}) \). Keeping (4.2) and (4.3) in hand, we compute

\[ \langle T_c(\tilde{A}) \ominus Z(\tilde{A}), Z(\tilde{A}) \rangle_\lambda = \left( \frac{(T_c(\tilde{A}), \tilde{N})_\lambda}{(\tilde{N}, \tilde{N})_\lambda} \ominus \tilde{N}, Z(\tilde{A}) \right)_\lambda = \frac{(T_c(\tilde{A}), \tilde{N})_\lambda}{(\tilde{N}, \tilde{N})_\lambda} \cdot \langle \tilde{N}, Z(\tilde{A}) \rangle_\lambda = 0, \]

since \( Z(\tilde{A}) \in \mathcal{Z} \). Hence, we obtain

\[ \langle T_c(\tilde{A}) \ominus Z(\tilde{A}), \tilde{X} \ominus Z(\tilde{A}) \rangle_\lambda = \langle T_c(\tilde{A}) \ominus Z(\tilde{A}), \tilde{X} \rangle_\lambda = \frac{(T_c(\tilde{A}), \tilde{N})_\lambda}{(\tilde{N}, \tilde{N})_\lambda} \cdot \langle \tilde{N}, \tilde{X} \rangle_\lambda. \]

Let \( \tilde{X} = (l, u, x, y)_\lambda \) be a trapezoidal fuzzy number. Proposition 2.3(1) implies

\[ \langle \tilde{N}, \tilde{X} \rangle_\lambda = l - u + (1 - \omega_L)x + (1 - \omega_U)y \leq 0. \]

According to the definition of \( \Gamma_2 \), we have \( \langle T_c(\tilde{A}), \tilde{N} \rangle_\lambda > 0 \). Hence, we conclude

\[ \langle T_c(\tilde{A}) \ominus Z(\tilde{A}), \tilde{X} \ominus Z(\tilde{A}) \rangle_\lambda \leq 0. \]

This completes the proof in this case.

Case (3) and (4): Let \( \tilde{A} \in \Gamma_3 \) and \( Z_x(\tilde{A}) = (l_x, u_x, x_x, 0) \). According to the definition of \( \mathcal{Z}_x \), we obtain

\[ l_x - u_x + (1 - \omega_L)x_x = 0. \]

To prove \( Z_x(\tilde{A}) \) is trapezoidal (or triangular), by Proposition 2.3 it suffices to claim \( x_x \geq 0 \). In Lemma 4.3, we show that \( Z_x(\tilde{A}) = T_c(\tilde{A}) \ominus N_x \). Substituting by (4.4), we obtain

\[ x_x = x_c - c^{-1}(1 - \omega_L) \cdot \frac{(T_c(\tilde{A}), \tilde{N})_\lambda - y_c(1 - \omega_U)}{(\tilde{N}, \tilde{N})_\lambda - d^{-1}(1 - \omega_U)^2}. \]

Consequently substituting by (5.3) and (5.4), we compute

\[ x_x = x_c - c^{-1}(1 - \omega_L) \cdot \frac{(1 - \omega_L)x_c - (u_c - l_c)}{a^{-1} + b^{-1} + c^{-1}(1 - \omega_L)^2} \]

\[ = \frac{(a^{-1} + b^{-1})x_c + c^{-1}(1 - \omega_L)(u_c - l_c)}{a^{-1} + b^{-1} + c^{-1}(1 - \omega_L)^2}. \]  

(5.8)

Since \( x_c \geq 0 \) (from Lemma 3.4) and \( u_c - l_c \geq 0 \) (from (3.5)), we obtain \( x_x \geq 0 \).

To prove \( T(\tilde{A}) = Z_x(\tilde{A}) \) for the case \( \tilde{A} \in \Gamma_3 \), it suffices to prove (5.5) by setting \( T(\tilde{A}) = Z_x(\tilde{A}) \). From (4.6), we find

\[ \langle \tilde{N}_x, Z_x(\tilde{A}) \rangle_\lambda = 0, \]

since \( Z_x(\tilde{A}) \in \mathcal{Z}_x \). Let \( \tilde{X} = (l, u, x, y)_\lambda \) be a trapezoidal fuzzy number. Applying Proposition 2.3(1) we obtain

\[ y \geq 0 \quad \text{and} \quad \langle \tilde{N}, \tilde{X} \rangle_\lambda = l - u + (1 - \omega_L)x + (1 - \omega_U)y \leq 0. \]
Now, let us compute from the left side of (5.5):
\[ \langle T_c(\tilde{A}) \ominus Z_x(\tilde{A}), \tilde{X} \ominus Z_x(\tilde{A}) \rangle_{\lambda} \]  (by Lemma 4.3)
\[ = \langle \tilde{N}_x, \tilde{X} \ominus Z_x(\tilde{A}) \rangle_{\lambda} \]
\[ = \langle \tilde{N}_x, \tilde{X} \rangle_{\lambda} \]  (by (4.4))
\[ = \frac{\langle T_c(\tilde{A}), \tilde{N} \rangle_{\lambda} - y_c(1 - \omega_U)}{\langle \tilde{N}, \tilde{N} \rangle_{\lambda} - d^{-1}(1 - \omega_U)^2} \cdot \langle \tilde{N}, \tilde{X} \rangle_{\lambda} + \frac{y_c(\tilde{N}, \tilde{N})_{\lambda} - d^{-1}(1 - \omega_U)\langle T_c(\tilde{A}), \tilde{N} \rangle_{\lambda}}{\langle \tilde{N}, \tilde{N} \rangle_{\lambda} - d^{-1}(1 - \omega_U)^2} \cdot dy. \]

Notice that
\[ \langle \tilde{N}, \tilde{N} \rangle_{\lambda} - d^{-1}(1 - \omega_U)^2 = a^{-1} + b^{-1} + e^{-1}(1 - \omega_L)^2 > 0. \]
The assumption \( \langle T_c(\tilde{A}), \tilde{N} \rangle_{\lambda} > \langle \tilde{N}, \tilde{N} \rangle_{\lambda} \cdot dy_c(1 - \omega_U)^{-1} \) implies that
\[ y_c(\tilde{N}, \tilde{N})_{\lambda} - d^{-1}(1 - \omega_U)(T_c(\tilde{A}), \tilde{N})_{\lambda} < 0 \]
and
\[ \langle T_c(\tilde{A}), \tilde{N} \rangle_{\lambda} - y_c(1 - \omega_U) > \langle \tilde{N}, \tilde{N} \rangle_{\lambda} \cdot dy_c(1 - \omega_U)^{-1} - y_c(1 - \omega_U) \]
\[ = dy_c(1 - \omega_U)^{-1}[(\tilde{N}, \tilde{N})_{\lambda} - d^{-1}(1 - \omega_U)^2] \]
\[ \geq 0, \]
since \( y_c \geq 0 \). By applying all of the above inequalities, we obtain
\[ \langle T_c(\tilde{A}) \ominus Z_x(\tilde{A}), \tilde{X} \ominus Z_x(\tilde{A}) \rangle_{\lambda} \leq 0. \]

This proves the case (3). In the same way, we can prove the other cases. \( \square \)

The weighted triangular approximation of a fuzzy number \( \tilde{A} \), denoted by \( \Delta(\tilde{A}) \), is the triangular fuzzy number that minimizes the distance \( d_{\lambda}(\tilde{A}, \tilde{X}) \), where \( \tilde{X} \in \mathbb{F} \) is triangular. In the preceding way (proof of Theorem 5.2), we can prove the following theorem.

**Theorem 5.3.** Let \( \tilde{A} \) be a fuzzy number, \( T_c(\tilde{A}) = (l_c, u_c, x_c, y_c)_{\lambda} \) be its weighted extended trapezoidal approximation, and let \( Z(\tilde{A}), Z_x(\tilde{A}), \) and \( Z_y(\tilde{A}) \) be the projections of \( T_c(\tilde{A}) \) onto the subspaces \( \mathcal{Z}, \mathcal{Z}_x, \) and \( \mathcal{Z}_y \), respectively. Then the weighted triangular approximation \( \Delta(\tilde{A}) \) can be calculated in the following cases:

1. \( \tilde{A} \in \Gamma_1 \cup \Gamma_2 \) implies \( \Delta(\tilde{A}) = Z(\tilde{A}), \)
2. \( \tilde{A} \in \Gamma_3 \) implies \( \Delta(\tilde{A}) = Z_x(\tilde{A}), \)
3. \( \tilde{A} \in \Gamma_4 \) implies \( \Delta(\tilde{A}) = Z_y(\tilde{A}). \)

**Proof.** From Proposition 2.3(2), we find that the set of all triangular fuzzy numbers is equal to
\[ \{(l, u, x, y)_{\lambda} \in \mathbb{C} | x, y \geq 0, l - u + (1 - \omega_L)x + (1 - \omega_U)y = 0\}, \]
hence it is closed and convex. Here, we only show that \( \tilde{A} \in \Gamma_1 \cup \Gamma_2 \) implies \( \Delta(\tilde{A}) = Z(\tilde{A}). \) Notice that
\[ \Gamma_1 \cup \Gamma_2 = \{ \tilde{A} \in \mathbb{F} | \langle T_c(\tilde{A}), \tilde{N} \rangle_{\lambda} \leq \langle \tilde{N}, \tilde{N} \rangle_{\lambda} \cdot \min\{c x_c(1 - \omega_L)^{-1}, d y_c(1 - \omega_U)^{-1}\} \}. \]  (5.9)

By applying (5.6) and (5.7) we obtain \( x_z, y_z \geq 0 \). Hence, \( Z(\tilde{A}) \) is triangular. From (4.2) and (4.3), we find
\[ \langle T_c(\tilde{A}) \ominus Z(\tilde{A}), \tilde{X} \rangle_{\lambda} = \frac{\langle T_c(\tilde{A}), \tilde{N} \rangle_{\lambda}}{\langle \tilde{N}, \tilde{N} \rangle_{\lambda}} \cdot \langle \tilde{N}, \tilde{X} \rangle_{\lambda} = 0 \]
for all \( \tilde{X} \in \mathcal{Z} \), so that
\[ \langle T_c(\tilde{A}) \ominus Z(\tilde{A}), \tilde{X} \ominus Z(\tilde{A}) \rangle_{\lambda} = 0. \]

By applying Fact 2.1, we obtain \( Z(\tilde{A}) \) is the weighted triangular approximation of \( \tilde{A}. \) \( \square \)
6. Algorithms and examples

Let $\lambda_L = \lambda_L(x)$ and $\lambda_U = \lambda_U(x)$ be two weighted functions on $[0,1]$, and $a, b, \omega_L, \omega_U, c, d$ be positive real numbers defined as (2.1)–(2.6). For arbitrary fuzzy number $\tilde{A} = [A_L(x), A_U(x)]$, we compute

$$L_0 := \int_0^1 A_L(x)\lambda_L(x) \, dx, \quad L_1 := \int_0^1 xA_L(x)\lambda_L(x) \, dx,$$

$$U_0 := \int_0^1 A_U(x)\lambda_U(x) \, dx, \quad U_1 := \int_0^1 xA_U(x)\lambda_U(x) \, dx. \quad (6.1)$$

By applying (3.1)–(3.4), throughout an elementary computation we will obtain the weighted extended trapezoidal approximation

$$T_{e_1}(\tilde{A}) = [t_1 - x_e(1 - x), t_2 + y_e(1 - x)],$$

where $t_1, t_2, x_e, y_e$ are computed by the following equalities:

$$t_1 = [a^{-1} - c^{-1}\omega_L(1 - \omega_L)]L_0 + c^{-1}(1 - \omega_L)L_1, \quad (6.3)$$

$$t_2 = [b^{-1} - d^{-1}\omega_U(1 - \omega_U)]U_0 + d^{-1}(1 - \omega_U)U_1, \quad (6.4)$$

$$x_e = -c^{-1}\omega_L L_0 + c^{-1}L_1, \quad (6.5)$$

$$y_e = d^{-1}\omega_U U_0 - d^{-1}U_1. \quad (6.6)$$

Consequently applying (5.3), we obtain

$$\langle T_{e_1}(\tilde{A}), \tilde{N} \rangle = t_1 - t_2.$$

Hence, applying Theorem 5.2 we find that if $t_1 \leq t_2$ then the weighted trapezoidal approximation is $T_{e_1}(\tilde{A})$, otherwise it is one of the three triangular fuzzy numbers: $Z(\tilde{A}), Z_\ell(\tilde{A})$, and $Z_\gamma(\tilde{A})$. Let

$$Z(\tilde{A}) = [t_2 - x_\gamma(1 - x), t_\gamma + y_\gamma(1 - x)].$$

By applying (4.3), throughout an elementary computation we will obtain

$$t_\ell = \frac{[b^{-1} + d^{-1}(1 - \omega_U)^2]t_1 + [a^{-1} + c^{-1}(1 - \omega_L)^2]t_2}{a^{-1} + b^{-1} + c^{-1}(1 - \omega_L)^2 + d^{-1}(1 - \omega_U)^2}, \quad (6.7)$$

$$x_\ell = x_e - \frac{c^{-1}(1 - \omega_L)(t_1 - t_2)}{a^{-1} + b^{-1} + c^{-1}(1 - \omega_L)^2 + d^{-1}(1 - \omega_U)^2}, \quad (6.8)$$

$$y_\ell = y_e - \frac{d^{-1}(1 - \omega_U)(t_1 - t_2)}{a^{-1} + b^{-1} + c^{-1}(1 - \omega_L)^2 + d^{-1}(1 - \omega_U)^2}. \quad (6.9)$$

While $\lambda_L = \lambda_U = \lambda(x)$ and $\int_0^1 \lambda(x) \, dx = \frac{1}{2}$, we obtain

$$a = b = \frac{1}{2}, \quad \omega_L = \omega_U = 2 \int_0^1 \lambda(x) \, dx,$$

$$c = d = \int_0^1 \lambda(x) \, dx - 2 \left[ \int_0^1 x\lambda(x) \, dx \right]^2,$$

and $t_\ell = \frac{1}{2}(t_1 + t_2)$. The readers can verify that, in this case (6.7)–(6.9) coincide with [21, Eq. (4)]. On the other hand, from (5.6) and (5.7) we can rewrite the four subsets $\Gamma_i$ (defined in Section 5), $1 \leq i \leq 4$, as follows:

$$\Gamma_1 := \{ \tilde{A} \in \mathbb{F} \mid t_1 \leq t_2 \},$$

$$\Gamma_2 := \{ \tilde{A} \in \mathbb{F} \mid t_1 > t_2, x_\ell \geq 0, y_\ell \geq 0 \},$$
\[
\Gamma_3 := \{ \tilde{A} \in \mathbb{F} \mid y_2 < 0 \}, \\
\Gamma_4 := \{ \tilde{A} \in \mathbb{F} \mid x_2 < 0 \}.
\]

Similarly, let \( Z_x(\tilde{A}) = [t_x - x(1 - z), t_x] \) and \( Z_y(\tilde{A}) = [t_y, t_y + y(1 - z)] \). From Lemma 4.3, we can find
\[
t_x = \frac{[a^{-1} - c^{-1} \omega_L(1 - \omega_L)]L_0 + c^{-1}(1 - \omega_L)L_1 + [a^{-1} + c^{-1}(1 - \omega_L)^2]U_0}{b[a^{-1} + b^{-1} + c^{-1}(1 - \omega_L)^2]}, \\
x_x = \frac{-(a^{-1} + b^{-1})L_0 + (a^{-1} + b^{-1})L_1 + b^{-1}(1 - \omega_L)U_0}{c[a^{-1} + b^{-1} + c^{-1}(1 - \omega_L)^2]},
\]
and
\[
t_y = \frac{[b^{-1} - d^{-1} \omega_U(1 - \omega_U)]U_0 + d^{-1}(1 - \omega_U)U_1 + [b^{-1} + d^{-1}(1 - \omega_U)^2]L_0}{a[a^{-1} + b^{-1} + d^{-1}(1 - \omega_U)^2]}, \\
y_y = \frac{(b^{-1} + a^{-1})U_0 - (a^{-1} + b^{-1})U_1 - a^{-1}(1 - \omega_U)L_0}{d[a^{-1} + b^{-1} + d^{-1}(1 - \omega_U)^2]}.
\]

Now, by applying Theorem 5.2 we propose the following algorithm for computing the weighted trapezoidal approximation \( T(\tilde{A}) \).

**Algorithm 1.** Input two weighted functions \( \lambda_L = \lambda_L(z) \) and \( \lambda_U = \lambda_U(z) \) on \([0, 1] \) and a fuzzy number \( \tilde{A} = [A_L(z), A_U(z)] \).

**Step 1.** Compute \( a, b, \omega_L, \omega_U, c, d \) by (2.1)–(2.6), \( L_0, L_1, U_0, U_1 \) by (6.1) and (6.2), and \( t_1, t_2, x_e, y_e \) by (6.3)–(6.6).

**Step 2.** If \( t_1 \leq t_2 \) then output \( T(\tilde{A}) = [t_1 - x_e(1 - z), t_2 + y_e(1 - z)] \) and stop.

**Step 3.** Compute \( x_z \) and \( y_z \) by (6.8) and (6.9).

**Step 4.** If \( x_z \geq 0 \) and \( y_z \geq 0 \) then compute \( t_z \) by (6.7). Output \( T(\tilde{A}) = [t_z - x_z(1 - z), t_z + y_z(1 - z)] \), and stop.

**Step 5.** If \( y_z < 0 \) then compute \( t_x \) and \( x_x \) by (6.10) and (6.11). Output \( T(\tilde{A}) = [t_x - x_x(1 - z), t_x] \), and stop.

**Step 6.** If \( x_z < 0 \) then compute \( y_y \) by (6.12) and (6.13). Output \( T(\tilde{A}) = [t_y, t_y + y_y(1 - z)] \), and stop.

**Example 6.1.** Let \( \lambda_L(z) = \lambda_U(z) = z \). Consider the weighted trapezoidal approximations of the following fuzzy numbers: (1) \( \tilde{A} = [-1 + \sqrt{z}, 1 - x^2] \), (2) \( \tilde{B} = [-1 + \sqrt{z}, 1 - \sqrt{z}] \), and (3) \( \tilde{C} = [-1 + \sqrt{z}, 0] \).

By applying (2.1)–(2.6), we get
\[
a = b = \frac{1}{2}, \quad \omega_L = \omega_U = \frac{2}{3} \quad \text{and} \quad c = d = \frac{1}{35}.
\]

(1) \( \tilde{A} = [-1 + \sqrt{z}, 1 - x^2] \). From (6.1) and (6.2), we compute
\[
L_0 = -\frac{1}{10}, \quad L_1 = -\frac{1}{11}, \quad U_0 = \frac{1}{4}, \quad U_1 = \frac{2}{17},
\]
so that by (6.3)–(6.6) we obtain
\[
t_1 = \frac{1}{35}, \quad t_2 = \frac{1}{10}, \quad x_e = \frac{24}{35}, \quad y_e = \frac{6}{5}.
\]

Since \( t_1 \leq t_2 \), by Step 2 we obtain (see Fig. 2)
\[
T(\tilde{A}) = [\frac{1}{35} - \frac{24}{35}(1 - z), \frac{1}{10} + \frac{6}{5}(1 - z)].
\]

(2) \( \tilde{B} = [-1 + \sqrt{z}, 1 - \sqrt{z}] \). From (6.1) and (6.2), we compute
\[
L_0 = -\frac{1}{10}, \quad L_1 = -\frac{1}{11}, \quad U_0 = \frac{1}{10}, \quad U_1 = \frac{1}{17},
\]
so that by (6.3)–(6.6) we obtain
\[
t_1 = \frac{1}{35}, \quad t_2 = -\frac{1}{35}, \quad x_e = \frac{24}{35}, \quad y_e = \frac{24}{35}.
\]
Since $t_1 > t_2$, go to Step 3. By (6.8) and (6.9), we compute

$$x_z = \frac{22}{35} \quad \text{and} \quad y_z = \frac{22}{35}.$$  

Since $x_z \geq 0$ and $y_z \geq 0$, go to Step 4. From (6.7) we find

$$t_z = \frac{t_1 + t_2}{2} = 0.$$  

Therefore (see Fig. 2)

$$T(\tilde{A}) = \left[ -\frac{22}{35}(1 - x), \frac{22}{35}(1 - x) \right].$$

(3) $\tilde{C} = [-1 + \sqrt{x}, 0]$. From (6.1) and (6.2), we compute

$$L_0 = -\frac{1}{10}, \quad L_1 = -\frac{1}{11}, \quad U_0 = 0, \quad U_1 = 0,$$

so that by (6.3)–(6.6) we obtain

$$t_1 = \frac{1}{35}, \quad t_2 = 0, \quad x_\epsilon = \frac{24}{35}, \quad y_\epsilon = 0.$$  

Since $t_1 > t_2$, go to Step 3. By (6.8) and (6.9), we compute

$$x_z = \frac{23}{35} \quad \text{and} \quad y_z = \frac{22}{35}.$$  

Since $y_z < 0$, go to Step 5. From (6.10) and (6.11) we find

$$t_x = \frac{1}{140} \quad \text{and} \quad x_x = \frac{9}{14}.$$  

Therefore (see Fig. 2)

$$T(\tilde{C}) = \left[ \frac{1}{140}, -\frac{9}{14}(1 - x), \frac{1}{140} \right].$$

In the same vein, by applying Theorem 5.3 we can also obtain the following algorithm.

**Algorithm 2.** Input two weighted functions $\lambda_L = \lambda_L(x)$ and $\lambda_U = \lambda_U(x)$ on $[0,1]$, and a fuzzy number $\tilde{A} = [A_L(x), A_U(x)]$.

**Step 1.** Compute $a, b, \omega_L, \omega_U, c, d$ by (2.1)–(2.6), $L_0, L_1, U_0, U_1$ by (6.1) and (6.2), $t_1, t_2, x_\epsilon, y_\epsilon$ by (6.3)–(6.6), and $x_z$ and $y_z$ by (6.8) and (6.9).

**Step 2.** If $x_z \geq 0$ and $y_z \geq 0$ then compute $t_x$ by (6.7). Output $\Delta(\tilde{A}) = [t_x - x_\epsilon(1 - x), t_x + y_\epsilon(1 - x)]$, and stop.

**Step 3.** If $y_z < 0$ then compute $t_x$ and $x_x$ by (6.10) and (6.11). Output $\Delta(\tilde{A}) = [t_x - x_\epsilon(1 - x), t_x]$, and stop.

**Step 4.** If $x_z < 0$ then compute $t_y$ and $y_y$ by (6.12) and (6.13). Output $\Delta(\tilde{A}) = [t_y, t_y + y_y(1 - x)]$, and stop.

**Example 6.2.** Let $\lambda_L(x) = 1 - x$ and $\lambda_U(x) = x$. Consider the weighted trapezoidal approximation $T(\tilde{A})$ and weighted triangular approximation $\Delta(\tilde{A})$ of the fuzzy number $\tilde{A} = [-1 + \sqrt{x}, 1 - x^2]$.  

---

Fig. 2. Weighted trapezoidal approximations under $\lambda_L(x) = \lambda_U(x) = x$.  

\[ \begin{align*}  
\text{Fig. 2. Weighted trapezoidal approximations under } & \lambda_L(x) = \lambda_U(x) = x. 
\end{align*} \]
First, we compute the weighted trapezoidal approximation $T(\tilde{A})$ of $\tilde{A}$. By applying (2.1)–(2.6), we get

$$a = b = \frac{1}{2}, \quad \omega_L = \frac{1}{4}, \quad \omega_U = \frac{2}{3} \quad \text{and} \quad c = d = \frac{1}{36}.$$  

From (6.1)–(6.6), we compute

$$L_0 = -\frac{7}{30}, \quad L_1 = -\frac{11}{210}, \quad U_0 = \frac{1}{4}, \quad U_1 = \frac{2}{15},$$

so that

$$t_1 = \frac{1}{7}, \quad t_2 = \frac{1}{11}, \quad x_e = \frac{32}{35}, \quad y_e = \frac{6}{5}.$$  

Since $t_1 > t_2$, go to Step 3 in Algorithm 1. By (6.8) and (6.9), we compute

$$x_z = \frac{61}{70} \quad \text{and} \quad y_z = \frac{33}{28}.$$  

Since $x_z \geq 0$ and $y_z \geq 0$, go to Step 4 in Algorithm 1. From (6.7) we find

$$t_z = \frac{31}{280}.$$

Therefore (see Fig. 3)

$$T(\tilde{A}) = \left[ \frac{31}{280} - \frac{61}{70}(1 - x), \frac{31}{280} + \frac{33}{28}(1 - x) \right].$$

Since $t_1 > t_2$ and $x_z$, $y_z \geq 0$, we get $\tilde{A} \in \Gamma_2$. Theorem 5.3 implies

$$\Delta(\tilde{A}) = Z(\tilde{A}) = T(\tilde{A}).$$

7. Properties

In [10], Grzegorzewski and Mrówka proposed many properties of trapezoidal approximations. Here, we will extend to the four approximations $Z$, $T_e$, $T$, and $T$. From (3.1)–(3.4), we find that these operators $l_e(\cdot)$, $u_e(\cdot)$, $x_e(\cdot)$ and $y_e(\cdot)$ are all linear, hence $T_e$ is linear. In Section 4, we claim that $Z(\tilde{A})$, $Z_x(\tilde{A})$ and $Z_y(\tilde{A})$ are the projections of $T_e(\tilde{A})$ onto the subspaces $Z$, $Z_x$ and $Z_y$, respectively. Hence, $Z(\tilde{A})$, $Z_x(\tilde{A})$ and $Z_y(\tilde{A})$ are linear, too.

**Proposition 7.1.** $T$ is linear on each of $\Gamma_i$, $1 \leq i \leq 4$, and $\Delta$ is linear on each of $\Gamma_1 \cup \Gamma_2$, $\Gamma_3$, and $\Gamma_4$.

**Proof.** In Section 5, we define $\Gamma_i$, $1 \leq i \leq 4$, by means of $T_e$, $x_e$, and $y_e$. For each $i$, if $\tilde{A}$ and $\tilde{B} \in \Gamma_i$, then

$$T_e(\tilde{A} + \tilde{B}) = T_e(\tilde{A}) + T_e(\tilde{B}), \quad x_e(\tilde{A} + \tilde{B}) = x_e(\tilde{A}) + x_e(\tilde{B}), \quad y_e(\tilde{A} + \tilde{B}) = y_e(\tilde{A}) + y_e(\tilde{B}),$$

where $\tilde{A}, \tilde{B} \in \Gamma_i$. Therefore, $T_e$ is linear. The same can be shown for $T$, $Z$, $Z_x$, and $Z_y$. Hence, $T$, $Z$, $Z_x$, and $Z_y$ are linear, respectively.
Proposition 7.3. Let $\tilde{A} + \tilde{B} \in \Gamma_1$. By applying Theorem 5.2, $T$ is linear on each of $\Gamma_i$, $1 \leq i \leq 4$. On the other hand, suppose that $\tilde{A}$ and $\tilde{B} \in \Gamma_1 \cup \Gamma_2$. By applying (5.9), we compute
\[
\langle T_\varepsilon(\tilde{A} + \tilde{B}), \tilde{N} \rangle \lambda = \langle T_\varepsilon(\tilde{A}), \tilde{N} \rangle \lambda + \langle T_\varepsilon(\tilde{B}), \tilde{N} \rangle \lambda
\]
\[
\leq \langle \tilde{N}, \tilde{N} \rangle \lambda \cdot \min\{cx_\varepsilon(\tilde{A})(1 - \omega_L)^{-1}, dy_\varepsilon(\tilde{A})(1 - \omega_U)^{-1}\}
\]
\[
\quad + \langle \tilde{N}, \tilde{N} \rangle \lambda \cdot \min\{cx_\varepsilon(\tilde{B})(1 - \omega_L)^{-1}, dy_\varepsilon(\tilde{B})(1 - \omega_U)^{-1}\}
\]
\[
\leq \langle \tilde{N}, \tilde{N} \rangle \lambda \cdot \min\{cx_\varepsilon(\tilde{A} + \tilde{B})(1 - \omega_L)^{-1}, dy_\varepsilon(\tilde{A} + \tilde{B})(1 - \omega_U)^{-1}\}.\]

Hence, we obtain $\tilde{A} + \tilde{B} \in \Gamma_1 \cup \Gamma_2$. By applying Theorem 5.3, $A$ is linear on each of $\Gamma_1 \cup \Gamma_2$, $\Gamma_3$, and $\Gamma_4$. \hfill \Box

**Proposition 7.2.** $Z, T_\varepsilon, A$, and $T$ all satisfy translation invariance and scale invariance.

**Proof.** Let $z \in \mathbb{R}$. From (3.1)--(3.4), we find $T_\varepsilon(z) = z$. Hence,
\[
T_\varepsilon(\tilde{A} + z) = T_\varepsilon(\tilde{A}) + T_\varepsilon(z) = T_\varepsilon(\tilde{A}) + z.
\]
Similarly, we also have
\[
Z(\tilde{A} + z) = Z(\tilde{A}) + z, \quad Z_x(\tilde{A} + z) = Z_x(\tilde{A}) + z, \quad Z_y(\tilde{A} + z) = Z_y(\tilde{A}) + z.
\]

Since $\tilde{A} \in \Gamma_i$ implies $\tilde{A} + z \in \Gamma_i$, by applying Theorems 5.2 and 5.3, $A$ and $T$ both satisfy translation invariance. Also, $\tilde{A} \in \Gamma_i$ implies $z\tilde{A} \in \Gamma_i$. By applying Proposition 7.1, they satisfy scale invariance. \hfill \Box

Recall that, a function $f = f(x)$ is called *Lipschitz continuous* if it satisfies
\[
d(f(x), f(y)) \leq d(x, y).
\]

Here, we show that:

**Proposition 7.3.** $Z, T_\varepsilon, A$, and $T$ are Lipschitz continuous.

**Proof.** Let $T_\varepsilon(\tilde{A}) = \{T_\varepsilon(\tilde{A})_L(x), T_\varepsilon(\tilde{A})_U(x)\}$ and $T_\varepsilon(\tilde{B}) = \{T_\varepsilon(\tilde{B})_L(x), T_\varepsilon(\tilde{B})_U(x)\}$ be $x$-cuts representations. From (3.6), we find
\[
\int_0^1 [A_L(x) - T_\varepsilon(\tilde{A})_L(x)][T_\varepsilon(\tilde{A})_L(x) - T_\varepsilon(\tilde{B})_L(x)]\lambda_L(x) \, dx = 0
\]
and
\[
\int_0^1 [B_L(x) - T_\varepsilon(\tilde{B})_L(x)][T_\varepsilon(\tilde{A})_L(x) - T_\varepsilon(\tilde{B})_L(x)]\lambda_L(x) \, dx = 0,
\]
since $T_\varepsilon(\tilde{A})_L(x) - T_\varepsilon(\tilde{B})_L(x)$ is equal to $a + b\lambda$, for some real numbers $a, b$. Applying the two equalities, we compute
\[
\int_0^1 [A_L(x) - B_L(x)]^2\lambda_L(x) \, dx
\]
\[
= \int_0^1 [(A_L(x) - T_\varepsilon(\tilde{A})_L(x)) - (B_L(x) - T_\varepsilon(\tilde{B})_L(x))]^2\lambda_L(x) \, dx
\]
\[
= \int_0^1 [(A_L(x) - T_\varepsilon(\tilde{A})_L(x)) - (B_L(x) - T_\varepsilon(\tilde{B})_L(x))]^2\lambda_L(x) \, dx.
\]
For determining the two approximations, an inner product on the space of all extended trapezoidal fuzzy numbers and its weighted 2-distance, so as to propose the improved weighted triangular approximation \( \Delta(\tilde{A}) \) and a weighted trapezoidal approximation \( T(\tilde{A}) \). For determining the two approximations, an inner product on the space of all extended trapezoidal fuzzy numbers and a weighted extended trapezoidal approximation \( T_e(\tilde{A}) \) are proposed. Finally, two efficient algorithms for computing \( T(\tilde{A}) \) and \( \Delta(\tilde{A}) \) are presented by means of dividing the set of all fuzzy numbers into four parts: \( \Gamma_i, 1 \leq i \leq 4 \). In fact, the four approximations \( Z, T_e, \Delta, \) and \( T \) all satisfy translation invariance, scale invariance, identity criterion. Moreover, we prove that

(1) \( Z \) and \( T_e \) are both linear on \( \mathbb{F} \),
(2) $T_c$ is linear on each of $\Gamma_i$, $1 \leq i \leq 4$, and

(3) $A$ is linear on each of $\Gamma_1 \cup \Gamma_2$, $\Gamma_3$, and $\Gamma_4$.

In addition, they all satisfy Lipschitz condition, that is

\begin{align*}
\tilde{d}_L(T_c(\tilde{A}), T_c(\tilde{B})) & \leq \tilde{d}_L(\tilde{A}, \tilde{B}), \\
\tilde{d}_L(Z(\tilde{A}), Z(\tilde{B})) & \leq \tilde{d}_L(\tilde{A}, \tilde{B}), \\
\tilde{d}_L(\Delta(\tilde{A}), \Delta(\tilde{B})) & \leq \tilde{d}_L(\tilde{A}, \tilde{B}), \\
\tilde{d}_L(T(\tilde{A}), T(\tilde{B})) & \leq \tilde{d}_L(\tilde{A}, \tilde{B}),
\end{align*}

for all fuzzy numbers $\tilde{A}$ and $\tilde{B}$ and weighted functions $\lambda_L = \lambda_L(x)$ and $\lambda_U = \lambda_U(x)$ on $[0,1]$.

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**References**


