On improving trapezoidal and triangular approximations of fuzzy numbers

Chi-Tsuen Yeh *

Department of Mathematics Education, National University of Tainan, 33, Sec. 2, Shu-Lin Street, 70005 Tainan, Taiwan

Received 1 May 2007; received in revised form 20 September 2007; accepted 20 September 2007

Available online 1 October 2007

Abstract

Recently, various researchers have proved that the approximations of fuzzy numbers may fail to be fuzzy numbers, such as the trapezoidal approximations of fuzzy numbers. In this paper, we show by an example that the weighted triangular approximation of fuzzy numbers, proposed by Zeng and Li, may lead to the same result. For filling the gap, improvements of trapezoidal and triangular approximations are proposed. The formulas for computing the two improved approximations are provided. Some properties of the two improved approximations are also proved.

Keywords: Fuzzy number; Trapezoidal approximation; Weighted triangular approximation

1. Introduction

Fuzzy numbers are often used in practice to represent uncertain or incomplete information. An interesting problem is to approximate general fuzzy numbers by means of trapezoidal or triangular ones, so as to simplify the calculations. Recently, there have been many research papers investigating the approximations of fuzzy numbers [1–4,6–13,15–17]. In this paper, we investigate improving trapezoidal and triangular approximations of fuzzy numbers.

For arbitrary fuzzy numbers $A$ and $B$ with $z$-cuts $[A_L(z), A_U(z)]$ and $[B_L(z), B_U(z)]$, respectively, the quantity

$$d(A, B) := \sqrt{\int_0^1 (|A_L(z) - B_L(z)|^2 + |A_U(z) - B_U(z)|^2)dz}, \quad (1.1)$$

is a distance between $A$ and $B$. The nearest trapezoidal (nearest triangular, symmetric triangular, interval) approximation of $A$ is defined as the trapezoidal (triangular, symmetric triangular, rectangle) fuzzy number which minimizes the distance $d(A, X)$, where $X$ is a trapezoidal (triangular, symmetric triangular, rectangle).
fuzzy number. Obviously, the interval approximation, proposed by Grzegorzewski [8], and the symmetric triangular approximation, proposed by Ma et al. [13], are fuzzy numbers. But, the Grzegorzewski and Mrówka’s trapezoidal approximation of a fuzzy number, named the extended trapezoidal approximation in this paper, may fail to be a fuzzy number, see [5,15], and so does the Abbasbandy and Asady’s approximation [2]. In Section 2, we show by an example that the weighted triangular approximation of a fuzzy number, proposed by Zeng and Li, may fail to be a fuzzy number. In Section 3, the space of extended trapezoidal fuzzy numbers is introduced [16]. We also present a formula for computing the extended trapezoidal approximation of fuzzy numbers preserving the expected interval, which is proposed by Grzegorzewski and Mrówka [9]. In Section 4, we introduce the nearest trapezoidal approximation of fuzzy numbers, which is an improvement of the former trapezoidal approximations. A formula for computing the nearest trapezoidal approximation is provided. In the same way, we propose the nearest triangular approximation of fuzzy numbers and its formula. In Section 5, we study the algorithms for computing the trapezoidal and triangular approximations using the $z$-cut representations. In Section 6, we show some properties of the two approximations.

2. A counter-example

An arbitrary fuzzy number $A$ can be represented by an ordered pair of left continuous functions $[A_L(z), A_U(z)]$, $0 \leq z \leq 1$, which satisfy the following conditions:

- $A_L$ is increasing on $[0, 1]$.
- $A_U$ is decreasing on $[0, 1]$.
- $A_L(1) \leq A_U(1)$.

A fuzzy number $A$ is trapezoidal iff (if and only if) its $z$-cuts are of the form $[a_1 + (a_2 - a_1)z, a_4 - (a_4 - a_3)z]$, where $a_i$ are real numbers (denoted by $\mathbb{R}$) with the constraint:

$$a_1 \leq a_2 \leq a_3 \leq a_4.$$  \hspace{1cm} (2.1)

The set of all trapezoidal fuzzy numbers is denoted by $\mathbb{T}$. A trapezoidal fuzzy number is called triangular iff $a_2 = a_3$ holds.

Recall that a weighted function $\lambda = \lambda(z)$, $z \in [0, 1]$, is a non-negative integrable function with $\int_0^1 \lambda(z)dz > 0$. Let $\hat{\lambda} = \lambda(z)$ be a weighted function with $\int_0^1 \hat{\lambda}(z)dz = \frac{1}{3}$. In [17], Zeng and Li proposed a weighted triangular approximation of fuzzy numbers. The weighted triangular approximation $[t_1 - (1 - z)t_2, t_1 + (1 - z)t_3]$ of a fuzzy number $A = [A_L(z), A_U(z)]$ can be computed by the following formulas:

$$t_1 = \frac{\int_0^1 \hat{\lambda}(z)(1 - z)dz \cdot \int_0^1 \hat{\lambda}(z)(1 - z)(A_L(z) + A_U(z))dz}{2\left(\int_0^1 \hat{\lambda}(z)(1 - z)dz\right)^2 - \int_0^1 \hat{\lambda}(z)(1 - z)^2 dz} - \int_0^1 \hat{\lambda}(z)(1 - z)^2 dz \cdot \int_0^1 \hat{\lambda}(z)(A_L(z) + A_U(z))dz}{2\left(\int_0^1 \hat{\lambda}(z)(1 - z)dz\right)^2 - \int_0^1 \hat{\lambda}(z)(1 - z)^2 dz}, \hspace{1cm} (2.2)$$

$$t_2 = \frac{t_1 \int_0^1 \hat{\lambda}(z)(1 - z)dz - \int_0^1 \hat{\lambda}(z)(1 - z)A_L(z)dz}{\int_0^1 \hat{\lambda}(z)(1 - z)^2 dz}, \hspace{1cm} (2.3)$$

$$t_3 = \frac{\int_0^1 \hat{\lambda}(z)(1 - z)A_U(z)dz - t_1 \int_0^1 \hat{\lambda}(z)(1 - z)dz}{\int_0^1 \hat{\lambda}(z)(1 - z)^2 dz}, \hspace{1cm} (2.4)$$

see [17, Eq. (4)]. The following Fact 2.2 shows that the weighted triangular approximation of $A = [0, 1 - \sqrt{2}]$ is not a fuzzy number, for any weighted function $\lambda$ with $\int_0^1 \hat{\lambda}(z)dz = \frac{1}{3}$. 
Lemma 2.1. Let $\rho$ be a weighted function on $[0, 1]$, and $\omega_\rho$ denote the following positive real number
\[
\omega_\rho := \frac{\int_0^1 \rho(x) \omega dx}{\int_0^1 \rho(x) dx}.
\] (2.5)

Then,
\[
\int_0^1 \rho(x)(x - \omega_\rho)g(x)dx < 0
\]
for all strictly decreasing function $g$.

Proof. Let $F = F(t)$ denote the function
\[
F(t) := \int_0^t \rho(x)(x - \omega_\rho)dx.
\]

Obviously, $F(0) = 0$. Eq. (2.5) implies $F(1) = 0$. On the other hand, fundamental theorem of calculus gives
\[
F'(t) = \rho(t)(t - \omega_\rho).\]
(2.6)

Hence, $F(t) \leq 0$ for each $t \in [0, \omega_\rho]$, and $F(t) \geq 0$ for each $t \in [\omega_\rho, 1]$. We conclude that $F(t) \leq 0$ for all $t \in [0, 1]$.

Let $g$ be a strictly decreasing function. By using integration by parts, we get
\[
\int_0^1 \rho(x)(x - \omega_\rho)g(x)dx = g(x)F(x)|_0^1 - \int_0^1 F(x)dg(x) = - \int_0^1 F(x)dg(x) \leq 0.
\]

Suppose, for a contradiction, the last equality holds. Obviously, $F = 0$ almost everywhere [14, p. 30], so that $\rho(t) = 0$ almost everywhere by Eq. (2.6), which contradicts $\int_0^1 \rho(x)dx > 0$. This completes the proof.  

Fact 2.2. The weighted triangular approximation of $A = [0, 1 - \sqrt{2}]$, computed by Eqs. (2.2)–(2.4), is not a fuzzy number, for any weighted function $\lambda$ with $\int_0^1 \lambda(x)dx = \frac{1}{\sqrt{2}}$.

Proof. Let $A = [0, 1 - \sqrt{2}]$. Substituting into Eq. (2.2) by $A_L(x) = 0$ and $A_U(x) = 1 - \sqrt{2}$, we get
\[
t_1 \left[ 2 \left( \int_0^1 \lambda(x)(1 - x)dx \right)^2 - \int_0^1 \lambda(x)(1 - x)^2 dx \right] = \int_0^1 \lambda(x)(1 - x)dx \cdot \int_0^1 \lambda(x)(1 - x)A_U(x)dx - \int_0^1 \lambda(x)(1 - x)^2 dx \cdot \int_0^1 \lambda(x)A_U(x)dx
\]
\[
= \int_0^1 \lambda(x)(1 - x)dx \cdot \int_0^1 \lambda(x)A_U(x)dx - \int_0^1 \lambda(x)(1 - x)dx \cdot \int_0^1 \lambda(x)A_U(x)dx
\]
\[
= \int_0^1 \lambda(x)(1 - x)dx \cdot \int_0^1 \frac{\lambda(x)(1 - x)dx}{1 + \sqrt{2}} - \int_0^1 \lambda(x)(1 - x)dx \cdot \int_0^1 \frac{\lambda(x)(1 - x)dx}{1 + \sqrt{2}}.
\]

By applying Lemma 2.1 to $\rho(x) = \lambda(x)(1 - x)$ and $g(x) = \frac{1}{1 + \sqrt{2}}$, we obtain
\[
t_1 \left[ 2 \left( \int_0^1 \lambda(x)(1 - x)dx \right)^2 - \int_0^1 \lambda(x)(1 - x)^2 dx \right] = - \int_0^1 \lambda(x)(1 - x)dx \cdot \int_0^1 \frac{\lambda(x)(1 - x)dx}{1 + \sqrt{2}}(x - \omega_\rho)dx > 0,
\]
where
\[
\omega_\rho = \frac{\int_0^1 \lambda(x)(1 - x)dx}{\int_0^1 \lambda(x)(1 - x)dx}.
\]
On the other hand, by Schwarz's inequality we get

\[
\left( \int_0^1 \lambda(x)(1-x) \, dx \right)^2 \leq \int_0^1 \left| \sqrt{\lambda(x)} \right|^2 \, dx \cdot \int_0^1 \left| \sqrt{\lambda(x)}(1-x) \right|^2 \, dx = \frac{1}{2} \int_0^1 \lambda(x)(1-x)^2 \, dx.
\]

It is easily verified that the above equality holds iff \( \lambda(x) = 0 \), which contradicts \( \int_0^1 \lambda(x) \, dx = \frac{1}{2} \). Hence,

\[
2 \left( \int_0^1 \lambda(x)(1-x) \, dx \right)^2 - \int_0^1 \lambda(x)(1-x)^2 \, dx < 0.
\]

This implies \( t_1 < 0 \). Substituting into Eq. (2.3) by \( A_\lambda(x) = 0 \), we get

\[
t_2 = t_1 \cdot \frac{\int_0^1 \lambda(x)(1-x) \, dx}{\int_0^1 \lambda(x)(1-x)^2 \, dx}.
\]

Hence, \( t_1 \) and \( t_2 \) have the same sign, i.e. \( t_2 \leq 0 \). By (2.1), it is easily seen that \( [t_1 - (1-x)t_2, t_1 + (1-x)t_3] \in \mathbb{T} \) if and only if \( t_2, t_3 \geq 0 \). This shows that the weighted triangular approximation of \( A = [0, 1-\sqrt{x}] \), computed by Eqs. (2.2)–(2.4), is not a fuzzy number. \( \square \)

3. Preliminaries and extended trapezoidal approximations

In Section 2, we show that the weighted triangular approximation of a fuzzy number may fail to be in \( \mathbb{T} \). The following work is to improve trapezoidal and triangular approximations of fuzzy numbers. In this paper, the form \([a + bx, c + dx]\), \(a, b, c, d \in \mathbb{R}\) (it may fail to satisfy (2.1)) is called an extended trapezoidal fuzzy number. The distance \( d(A, B) \) between two extended trapezoidal fuzzy numbers \( A = [A_\lambda(x), A_\mu(x)] \) and \( B = [B_\lambda(x), B_\mu(x)] \) is defined as Eq. (1.1).

Let the symbol \((l, u, x, y)\) denote an extended trapezoidal fuzzy number with the following \( x \)-cuts

\[
\left[ l + x\left( x - \frac{1}{2} \right), y - y\left( x - \frac{1}{2} \right) \right].
\]

In [16], the author proved the following proposition.

**Proposition 3.1** [16, Proposition 3.1]. Let \( A = (l, u, x, y) \) be an extended trapezoidal fuzzy number. Then

1. \( A \) is trapezoidal iff \( x, y \geq 0 \) and \( x + y \leq 2(u - l) \).
2. \( A \) is triangular iff \( x, y \geq 0 \) and \( x + y = 2(u - l) \).

**Proof.** See Appendix A. \( \square \)

**Proposition 3.2** [16, Proposition 3.3]. Let \( A = (l_1, u_1; x_1, y_1) \) and \( B = (l_2, u_2; x_2, y_2) \) be two extended trapezoidal fuzzy numbers. Then

\[
d(A, B)^2 = (l_1 - l_2)^2 + (u_1 - u_2)^2 + \frac{1}{12} [(x_1 - x_2)^2 + (y_1 - y_2)^2].
\]

**Proof.** See Appendix B. \( \square \)

The extended trapezoidal approximation \( T_e = (l_e, u_e; x_e, y_e) \) of fuzzy number \( A \), which coincides with Grzegorzewski and Mrówka’s trapezoidal approximation under the preservation of the expected interval [9], is the extended trapezoidal fuzzy number which minimizes the distance \( d(A, X) \), where \( X \) is an extended trapezoidal fuzzy number. Grzegorzewski and Mrówka provided the following formulas for computing \( T_e(A) = [a_1 + (a_2 - a_1)x, a_4 - (a_4 - a_3)x] \), where \( a_i \) are defined as follows
\[
\begin{align*}
    a_1 &= -6 \int_0^1 A_L(x) \, dx + 4 \int_0^1 A_L(x) \, dx, \\
    a_2 &= 6 \int_0^1 A_L(x) \, dx - 2 \int_0^1 A_L(x) \, dx, \\
    a_3 &= 6 \int_0^1 A_U(x) \, dx - 2 \int_0^1 A_U(x) \, dx, \\
    a_4 &= -6 \int_0^1 A_U(x) \, dx + 4 \int_0^1 A_U(x) \, dx,
\end{align*}
\]

(3.2) (3.3) (3.4) (3.5)

see [9, Eqs. (29)–(32)]. Obviously (3.1) implies
\[
    l_e = \frac{a_1 + a_2}{2}, \quad u_e = \frac{a_3 + a_4}{2}, \quad x_e = a_2 - a_1, \quad y_e = a_4 - a_3.
\]

Hence, \( T_e(A) = (l_e, u_e; x_e, y_e) \) can be computed by the following formulas:
\[
\begin{align*}
    l_e &= \int_0^1 A_L(x) \, dx, \\
    u_e &= \int_0^1 A_U(x) \, dx, \\
    x_e &= 12 \int_0^1 \left( x - \frac{1}{2} \right) A_L(x) \, dx, \\
    y_e &= -12 \int_0^1 \left( x - \frac{1}{2} \right) A_U(x) \, dx.
\end{align*}
\]

(3.6) (3.7) (3.8) (3.9)

Obviously, the definition of fuzzy number \( A \) implies
\[
    u_e - l_e \geq 0. \quad (3.10)
\]

Because \( A_L(x) \) and \( A_U(x) \) are increasing and decreasing respectively, applying Lemma 2.1 to \( \rho = 1 \) (then \( \omega_p = \frac{1}{2} \)) we obtain
\[
    x_e \geq 0 \quad \text{and} \quad y_e \geq 0. \quad (3.11)
\]

In [15], the author proved two distance properties for \( T_e \).

**Proposition 3.3** [15, Proposition 4.2]. Let \( A \) be a fuzzy number. Then,
\[
    d(A, B)^2 = d(A, T_e(A))^2 + d(T_e(A), B)^2,
\]
for any extended trapezoidal fuzzy number \( B \).

**Proposition 3.4** [15, Proposition 4.4]. \( d(T_e(A), T_e(B)) \leq d(A, B) \), for all fuzzy numbers \( A, B \).

4. Nearest trapezoidal and triangular approximations

Let \( A \) be a fuzzy number, and \( T_n(A) \) denote the nearest trapezoidal approximation of \( A \), which means \( T_n(A) \) is the trapezoidal fuzzy number minimizing the distance \( d(A, X) \), where \( X \in \mathbb{T} \).

Let \( B = (l, u; x, y) \) be an arbitrary trapezoidal fuzzy number, and \( T_e(A) = (l_e, u_e; x_e, y_e) \) be the extended trapezoidal approximation of \( A \). **Proposition 3.2** gives
\[
    d(T_e(A), B)^2 = (l - l_e)^2 + (u - u_e)^2 + \frac{1}{12} [(x - x_e)^2 + (y - y_e)^2].
\]

By **Proposition 3.3**, we obtain
\[
    d(A, B)^2 = d(A, T_e(A))^2 + (l - l_e)^2 + (u - u_e)^2 + \frac{1}{12} [(x - x_e)^2 + (y - y_e)^2].
\]
Because $d(A, T_e(A))^2$ is constant, to minimize $d(A, B)$ it suffices to minimize the following function

$$f(l, u, x, y) := (l - l_e)^2 + (u - u_e)^2 + \frac{1}{12} [(x - x_e)^2 + (y - y_e)^2]. \tag{4.1}$$

**Proposition 3.1(1)** gives that $B = (l, u; x, y) \in T$ iff $x, y \geq 0$ and $x + y \leq 2(u - l)$. Hence, the problem of calculation of $T_e(A)$ can be replaced with the following problem.

**Problem 4.1.** Find $(l, u, x, y) \in \Omega$ that minimizes the function $f(l, u, x, y)$, defined as Eq. (4.1), where

$$\Omega := \{(l, u, x, y)| x \geq 0 \text{ and } x + y \leq 2(u - l)\}. \tag{4.2}$$

**Lemma 4.2.** Every solution $(l, u, x, y)$ of **Problem 4.1** satisfies $u_e - l_e \leq u - l$.

**Proof.** Suppose, for a contradiction, $(l, u, x, y)$ is a solution of **Problem 4.1** with $u - l < u_e - l_e$. Then the point $(l_e, u_e, x, y)$ is in $\Omega$, since

$$x, y \geq 0 \text{ and } x + y \leq 2(u - l) < 2(u_e - l_e).$$

Notice that

$$f(l, u, x, y) = (l - l_e)^2 + (u - u_e)^2 + \frac{1}{12} [(x - x_e)^2 + (y - y_e)^2] > \frac{1}{12} [(x - x_e)^2 + (y - y_e)^2] = f(l_e, u_e, x, y).$$

Hence, $f(l, u, x, y)$ is not minimal, which is a contradiction. \square

**Lemma 4.3.** If $x_e + y_e \geq 2(u_e - l_e)$, then every solution $(l, u, x, y)$ of **Problem 4.1** satisfies $x + y = 2(u - l)$.

**Proof.** Suppose, for a contradiction, $(l, u, x, y)$ is a solution of **Problem 4.1** with $x + y < 2(u - l)$. If

$$x + y < 2(u_e - l_e),$$

then let

$$x' = x + \frac{2(u_e - l_e) - x - y}{2} \quad \text{and} \quad y' = y + \frac{2(u_e - l_e) - x - y}{2}.$$

Obviously, we have $x' \geq x$, $y' \geq y$ and

$$\begin{cases} x' + y' = 2(u_e - l_e), \\ x' - y' = x - y. \end{cases}$$

By applying **Lemma 4.2**, the first equation implies $x' + y' \leq 2(u - l)$. Hence $(l, u, x', y') \in \Omega$, where $\Omega$ is defined as Eq. (4.2). Notice that

$$f(l, u, x', y') - [(l - l_e)^2 + (u - u_e)^2] = \frac{1}{12} [(x' - x_e)^2 + (y' - y_e)^2]
= \frac{1}{24} [(x' + y' - x_e - y_e)^2 + (x' - y' - (x_e - y_e))^2]
= \frac{1}{24} [(2(u_e - l_e) - x_e - y_e)^2 + (x - y - (x_e - y_e))^2].$$

Because $x_e + y_e \geq 2(u_e - l_e)$ and $x + y < 2(u_e - l_e)$, we have

$$0 \geq 2(u_e - l_e) - x_e - y_e \geq x + y - x_e - y_e.$$

This implies

$$(2(u_e - l_e) - x_e - y_e)^2 < (x + y - x_e - y_e)^2.$$
Let’s continue the preceding computation:

\[ f(l, u, x', y') - [(l - l_e)^2 + (u - u_e)^2] < \frac{1}{24} [(x + y - x_e - y_e)^2 + (x - y - (x_e - y_e))^2] \]

\[ = \frac{1}{12} [(x - x_e)^2 + (y - y_e)^2] = f(l, u, x, y) - [(l - l_e)^2 + (u - u_e)^2]. \]

Hence, \( f(l, u, x', y') < f(l, u, x, y) \), which is a contradiction.

If \( x + y \geq 2(u_e - l_e) \), then let

\[ l' = l + \frac{2(u - l) - x - y}{4} \quad \text{and} \quad u' = u - \frac{2(u - l) - x - y}{4}. \]

We obtain

\[ 2(u' - l') = x + y \quad \text{and} \quad u' + l' = u + l. \]

The first equation implies \((l', u', x, y) \in \Omega\). Additionally, the assumptions \( x + y < 2(u - l) \) and \( x + y \geq 2(u_e - l_e) \) imply

\[ u - l > \frac{x + y}{2} \geq u_e - l_e, \]

so that

\[ [(u - l) - (u_e - l_e)]^2 > \left[ \frac{x + y}{2} - (u_e - l_e) \right]^2. \]

Now, let’s compute

\[ f(l', u', x, y) - \frac{1}{12} [(x - x_e)^2 + (y - y_e)^2] = (l' - l_e)^2 + (u' - u_e)^2 \]

\[ = \frac{1}{2} [(u' + l' - (u_e + l_e))^2 + (u' - l' - (u_e - l_e))^2] \]

\[ = \frac{1}{2} \left[ (u + l - (u_e + l_e))^2 + \left( \frac{x + y}{2} - (u_e - l_e) \right)^2 \right] \]

\[ < \frac{1}{2} [(u + l - (u_e + l_e))^2 + (u - l - (u_e - l_e))^2] \]

\[ = f(l, u, x, y) - \frac{1}{12} [(x - x_e)^2 + (y - y_e)^2]. \]

This also leads to a contradiction and completes the proof. □

**Theorem 4.4.** Let \( A = [A_L(x), A_R(x)] \) be a fuzzy number, \( T_e(A) = (l_e, u_e; x_e, y_e) \) be the extended trapezoidal approximation of \( A \), and

\[ \delta_e := x_e + y_e - 2(u_e - l_e). \]

Then the nearest trapezoidal approximation \( T_n(A) = (l_n, u_n; x_n, y_n) \) of \( A \) can be calculated in the following cases:

1. If \( \delta_e \leq 0 \), then \( T_n(A) = T_e(A) \).
2. If \( \delta_e > 0 \) and \( x_e, y_e \geq \frac{3}{8} \delta_e \), then

\[ l_n = l_e - \frac{1}{16} \delta_e, \quad u_n = u_e + \frac{1}{16} \delta_e, \quad x_n = x_e - \frac{3}{8} \delta_e, \quad y_n = y_e - \frac{3}{8} \delta_e. \]
Proof

(1) Let \( \delta_c \leq 0 \), i.e. \( x_c + y_c \leq 2(u_c - l_c) \). (3.11) gives \( x_c, y_c \geq 0 \). Hence, \((l_c, u_c, x_c, y_c) \in \Omega\), where \( \Omega \) is defined as Eq. (4.2). That implies \( T_n(A) = T_c(A) \).

(2) Let \( \delta_c > 0 \). By Lemma 4.3, the solution of Problem 4.1 satisfies \( x + y = 2(u - l) \). Applying Schwarz’s inequality we get

\[
(l - l_c)^2 + (u - u_c)^2 + \frac{1}{12} (x - x_c)^2 + \frac{1}{12} (y - y_c)^2 \geq [x + y - 2(u - l) - (x_c + y_c - 2(u_c - l_c))]^2 = \delta_c^2.
\]

The equality holds iff

\[
\frac{l - l_c}{2} = \frac{-(u - u_c)}{2} = \frac{x - x_c}{12} = \frac{y - y_c}{12} = \frac{-\delta_c}{4 + 4 + 12 + 12}.
\]

Hence we obtain Eq. (4.3). Because \( x_c, y_c \geq \frac{5}{8} \delta_c \). Eq. (4.3) implies \( x_n, y_n \geq 0 \). Additionally, \( x_n + y_n = 2(u_n - l_n) \). We obtain \((l_n, u_n, x_n, y_n) \in \Omega\). This completes the proof of the case.

(3) Because the solution \((l, u, x, y)\) of Problem 4.1 is in the boundary of \( \Omega \), by (2), the assumption \( x_c < \frac{5}{8} \delta_c \) implies \( x = 0 \) or \( y = 0 \). Suppose that \( y = 0 \). Note that (3.10) implies

\[
y_c \geq y_c - 2(u_c - l_c) = \delta_c - x_c > \frac{5}{8} \delta_c > x_c.
\]

Now, applying the above inequality we compute

\[
f(l, u, x, 0) = (l - l_c)^2 + (u - u_c)^2 + \frac{1}{12} [x - x_c]^2 + \frac{1}{12} [y - y_c]^2 = (l - l_c)^2 + (u - u_c)^2 + \frac{1}{12} [x^2 - 2xx_c + x_c^2 + y_c^2]
\]

\[
> (l - l_c)^2 + (u - u_c)^2 + \frac{1}{12} [x^2 - 2xy_c + y_c^2] = f(l, u, 0, x).
\]

Hence, \( f(l, u, x, 0) \) is not minimal, which is a contradiction. We conclude \( x = 0 \). By Lemma 4.3, we obtain \( y = 2u - 2l \). Substituting into Eq. (4.1) by \( x = 0 \) and \( y = 2u - 2l \), we get

\[
f(l, u, 0, 2u - 2l) = (l - l_c)^2 + (u - u_c)^2 + \frac{1}{12} x_c^2 + \frac{1}{12} (2u - 2l - y_c)^2.
\]

By applying elementary calculus, let’s solve the following equations:

\[
\begin{align*}
\frac{\partial f(l, u, 0, 2u - 2l)}{\partial x} &= 2(l - l_c) - \frac{1}{2}(2u - 2l - y_c) = 0, \\
\frac{\partial f(l, u, 0, 2u - 2l)}{\partial u} &= 2(u - u_c) + \frac{1}{2}(2u - 2l - y_c) = 0.
\end{align*}
\]

Through a mathematical calculation, we may obtain Eq. (4.4).

(4) In the same vein, we may also obtain Eq. (4.5).
Remark 4.5. (3.10) gives \( u_e - l_e \geq 0 \), so that

\[
x_e + y_e \geq x_e + y_e - 2(u_e - l_e) = \delta_e.
\]

This implies the intersection of Cases (3) and (4) is empty. Thus, the four cases are disjointed.

Remark 4.6. If \( \delta_e \geq 0 \), i.e. \( x_e + y_e \geq 2(u_e - l_e) \), Lemma 4.3 implies \( x_n + y_n = 2(u_n - l_n) \). By Proposition 3.1(2), \( T_n(A) = (l_n, u_n, x_n, y_n) \) is triangular. That shows the nearest trapezoidal approximations obtained in Cases (2)–(4) are all triangular.

Let \( A \) be a fuzzy number, and \( \Delta(A) \) denote the nearest triangular approximation of \( A \), which means \( \Delta(A) \) is the triangular fuzzy number minimizing the distance \( d(A, \Delta) \), where \( \Delta \) is a triangular fuzzy number. Similarly, by applying Proposition 3.1(2), the problem of calculation of \( \Delta(A) \) can be replaced with the following problem.

Problem 4.7. Find \((l, u, x, y)\) that minimizes \( f(l, u, x, y) \) with the constraints \( x, y \geq 0 \) and \( x + y = 2(u - l) \), where \( f(l, u, x, y) \) is defined as Eq. (4.1).

Using the techniques from the proof of Theorem 4.4, we obtain the following theorem.

Theorem 4.8. Let \( A = [A_L(x), A_U(x)] \) be a fuzzy number, \( T_e(A) = (l_e, u_e, x_e, y_e) \) be the extended trapezoidal approximation of \( A \), and

\[
\delta_e := x_e + y_e - 2(u_e - l_e).
\]

Then the nearest triangular approximation \( \Delta(A) = (l', u'; x', y') \) can be calculated in the following cases:

1. If \( x_e, y_e \geq \frac{3}{8}\delta_e \), then

\[
\begin{align*}
l' &= l_e - \frac{1}{16}\delta_e, \\
u' &= u_e + \frac{1}{16}\delta_e, \\
x' &= x_e - \frac{3}{8}\delta_e, \\
y' &= y_e - \frac{3}{8}\delta_e.
\end{align*}
\]

2. If \( x_e < \frac{3}{8}\delta_e \), then

\[
\begin{align*}
l' &= \frac{4}{5}l_e + \frac{1}{3}u_e - \frac{1}{10}y_e, \\
u' &= \frac{1}{5}l_e + \frac{4}{3}u_e + \frac{1}{10}y_e, \\
x' &= 0, \\
y' &= -\frac{6}{5}l_e + \frac{6}{5}u_e + \frac{2}{5}y_e.
\end{align*}
\]

3. If \( y_e < \frac{3}{8}\delta_e \), then

\[
\begin{align*}
l' &= \frac{4}{5}l_e + \frac{1}{3}u_e - \frac{1}{10}x_e, \\
u' &= \frac{1}{5}l_e + \frac{4}{3}u_e + \frac{1}{10}x_e, \\
x' &= -\frac{6}{5}l_e + \frac{6}{5}u_e + \frac{2}{5}x_e, \\
y' &= 0.
\end{align*}
\]

5. Algorithms and examples

In the previous section, we propose a method for computing the representations \((l, u, x, y)\) of the nearest trapezoidal and triangular approximations (Theorems 4.4 and 4.8). By applying (3.1), we will obtain their \( x \)-cut representations. In the following, we propose the algorithms for computing the \( x \)-cut representations of nearest trapezoidal and triangular approximations of fuzzy numbers.

Let \( A = [A_L(x), A_U(x)] \) be a fuzzy number, and define

\[
\phi := \int_0^1 (3x - 1)[A_U(x) - A_L(x)] dx.
\]
By applying Eqs. (3.6)–(3.9), we obtain
\[ \delta_e = x_e + y_e - 2(u_e - l_e) = -4\phi. \]

**Remark 4.6** implies that the nearest trapezoidal approximation in Case (2) of **Theorem 4.4** is triangular. Let \( T^*(A) = [t_1 - (1 - z)t_2, t_1 + (1 - z)t_3] \). By applying (3.1) and (4.3), through an elementary computation we obtain
\[
\begin{align*}
t_1 &= l_e + \frac{1}{2}x_e - \frac{1}{4}\delta_e = \int_0^1 (3x - 1)[A_L(x) + A_U(x)]dx, \\
t_2 &= x_e - \frac{3}{8}\delta_e = \frac{3}{2}t_1 + 3\int_0^1 (x - 1)A_L(x)dx, \\
t_3 &= y_e - \frac{3}{8}\delta_e = -\frac{3}{2}t_1 - 3\int_0^1 (x - 1)A_U(x)dx.
\end{align*}
\]

Notice that \( x_e < \frac{3}{8}\delta_e \) iff \( t_2 < 0 \) and \( y_e < \frac{3}{8}\delta_e \) iff \( t_3 < 0 \). Similarly, we may obtain \( T_n(A) = [s_1, s_1 + (1 - z)s_2] \) in Case (3) of **Theorem 4.4**, where
\[
\begin{align*}
s_1 &= \frac{4}{5} \int_0^1 A_L(x)dx + \frac{2}{5} \int_0^1 (3x - 1)A_U(x)dx, \\
s_2 &= -4s_1 + 2\int_0^1 (A_L(x) + A_U(x))dx,
\end{align*}
\]
and \( T_n(A) = [s_1 - (1 - z)s_2, s_1] \) in Case (4) of **Theorem 4.4**, where
\[
\begin{align*}
s_1 &= \frac{4}{5} \int_0^1 A_U(x)dx + \frac{2}{5} \int_0^1 (3x - 1)A_L(x)dx, \\
s_2 &= 4s_1 - 2\int_0^1 (A_L(x) + A_U(x))dx.
\end{align*}
\]

By applying **Theorem 4.4**, we propose the following algorithm.

**Algorithm 1.** Let \( A = [A_L(x), A_U(x)] \) be a fuzzy number, and \( T_n(A) \) be the nearest trapezoidal approximation of \( A \):

**Step 1.** Compute \( \phi \) by Eq. (5.1). If \( \phi \geq 0 \), then apply Grzegorzewski and Mrówka’s formulas (Eqs. (3.2)–(3.5)) to determine \( T_n(A) \).

**Step 2.** If \( \phi < 0 \), then compute \( t_1, t_2 \) and \( t_3 \) by Eq. (5.2). If \( t_2 \geq 0 \) and \( t_3 \geq 0 \), then \( T_n(A) = [t_1 - (1 - z)t_2, t_1 + (1 - z)t_3] \).

**Step 3.** If \( t_2 < 0 \), then \( T_n(A) = [s_1, s_1 + (1 - z)s_2] \), where \( s_1 \) and \( s_2 \) are computed by Eqs. (5.5) and (5.6).

**Step 4.** If \( t_3 < 0 \), then \( T_n(A) = [s_1 - (1 - z)s_2, s_1] \), where \( s_1 \) and \( s_2 \) are computed by Eqs. (5.7) and (5.8).

**Example 5.1.** Let’s consider the nearest trapezoidal approximations \( T_n \) of the following fuzzy numbers: (1) \( A = [-1 + z^2, 1 - \sqrt{z}] \), (2) \( B = [-1 + \sqrt{z}, 1 - \sqrt{z}] \), and (3) \( C = [0, 1 - \sqrt{z}] \).

(1) \( A = [-1 + z^2, 1 - \sqrt{z}] \). By applying Eq. (5.1), we get
\[ \phi(A) = \int_0^1 (3x - 1)(2 - x^2 - \sqrt{z})dx = \frac{1}{20} > 0. \]

Therefore, by Eqs. (3.2)–(3.5) (in Step 1) we obtain
\[ T_n(A) = \left[ -\frac{7}{6} + z, \frac{11}{15} - \frac{4}{5}z \right]. \]
(2) \( B = [-1 + \sqrt{a}, 1 - \sqrt{a}] \). By applying Eq. (5.1), we get
\[
\phi(B) = \int_0^1 (3a - 1)(2 - 2\sqrt{a})dx = \frac{-1}{15} < 0.
\]
Therefore, by Eqs. (5.2)–(5.4) in Step 2 we obtain \( t_1 = 0 \) and \( t_2 = t_3 = \frac{7}{10} > 0 \). Thus,
\[
T_a(B) = \left[ -\frac{7}{10}(1 - a), \frac{7}{10}(1 - a) \right].
\]
(3) \( C = [0, 1 - \sqrt{a}] \). By applying Eq. (5.1), we get
\[
\phi(C) = \int_0^1 (3a - 1)(1 - \sqrt{a})dx = \frac{-1}{30} < 0.
\]
Next, by applying Eqs. (5.2)–(5.4) in Step 2 we obtain \( t_1 = -\frac{1}{30}, t_2 = -\frac{1}{30}, \) and \( t_3 = \frac{3}{4} \). Since \( t_2 < 0 \), by Eqs. (5.5) and (5.6) in Step 3 we obtain \( s_1 = -\frac{1}{75} \) and \( s_2 = \frac{18}{25} \). Thus,
\[
T_a(C) = \left[ -\frac{1}{75}, \frac{53}{75} - \frac{18}{25} a \right].
\]

In the same vein, we may also obtain the following algorithm for computing the nearest triangular approximation of a fuzzy number.

**Algorithm 2.** Let \( A = [A_U(z), A_L(z)] \) be a fuzzy number, and \( \Delta(A) \) be the nearest triangular approximation of \( A \).

**Step 1.** Compute \( t_1, t_2 \) and \( t_3 \) by Eqs. (5.2)–(5.4). If \( t_2 \geq 0 \) and \( t_3 \geq 0 \), then \( \Delta(A) = [t_1 - (1 - \alpha)t_2, t_1 + (1 - \alpha)t_3] \).

**Step 2.** If \( t_2 < 0 \), then \( \Delta(A) = [s_1, s_1 + (1 - \alpha)s_2] \), where \( s_1 \) and \( s_2 \) are computed by Eqs. (5.5) and (5.6).

**Step 3.** If \( t_3 < 0 \), then \( \Delta(A) = [s_1 - (1 - \alpha)s_2, s_1] \), where \( s_1 \) and \( s_2 \) are computed by Eqs. (5.7) and (5.8).

**Example 5.2.** Let’s consider the nearest triangular approximation \( \Delta \) of the fuzzy number \( A = [-1 + \sqrt{a}, 1 - \sqrt{a}] \).

By applying Eqs. (5.2)–(5.4), we will obtain \( t_1 = -\frac{7}{60}, t_2 = \frac{43}{40}, \) and \( t_3 = \frac{7}{8} \). Because \( t_2 > 0 \) and \( t_3 > 0 \), by Step 1 we obtain
\[
\Delta(A) = \left[ -\frac{7}{60} - \frac{43}{40} (1 - \alpha), -\frac{7}{60} + \frac{7}{8} (1 - \alpha) \right].
\]

6. Properties

For arbitrary fuzzy numbers \( A = [A_U(z), A_L(z)], B = [B_U(z), B_L(z)] \) and a real number \( r \), the addition and scalar multiplication of fuzzy numbers can be represented as follows:

\[
A + B = [A_L(z) + B_L(z), A_U(z) + B_U(z)],
\]
\[
rA = \begin{cases} [rA_L(z), rA_U(z)] & \text{if } r \geq 0, \\ [rA_L(z), rA_U(z)] & \text{if } r < 0. \end{cases}
\]

Obviously, if \( A = (l_1, u_1; x_1, y_1) \) and \( B = (l_2, u_2; x_2, y_2) \), it is easily verified that
\[
A + B = (l_1 + l_2, u_1 + u_2; x_1 + x_2, y_1 + y_2), \quad \text{and} \quad (6.1)
\]
\[
rA = \begin{cases} (rl_1, ru_1; rx_1, ry_1) & \text{if } r \geq 0, \\ (ru_1, rl_1; -ry_1, -rx_1) & \text{if } r < 0. \end{cases} \quad \text{and} \quad (6.2)
\]
Let $T_c(A) = (l_c(A), u_c(A); x_c(A), y_c(A))$ be the extended trapezoidal approximation of fuzzy number $A$. In [15, Proposition 2.1], the author proved that

$$T_c(rA + sB) = rT_c(A) + sT_c(B),$$

(6.3)

for all fuzzy numbers $A$, $B$ and $r, s \in \mathbb{R}$. Hence, we have

$$l_c(A + B) = l_c(A) + l_c(B), \quad u_c(A + B) = u_c(A) + u_c(B),$$

(6.4)

and

$$x_c(A + B) = x_c(A) + x_c(B), \quad y_c(A + B) = y_c(A) + y_c(B),$$

(6.5)

and

(1) if $r \geq 0$, then

$$l_c(rA) = rl_c(A), \quad u_c(rA) = ru_c(A), \quad x_c(rA) = rx_c(A), \quad y_c(rA) = ry_c(A),$$

(6.6)

(2) if $r < 0$, then

$$l_c(rA) = ru_c(A), \quad u_c(rA) = rl_c(A), \quad x_c(rA) = -ry_c(A), \quad y_c(rA) = -rx_c(A).$$

(6.7)

The above equalities may also be verified by applying Eqs. (3.6)–(3.9). Let

$$\delta_c(A) := x_c(A) + y_c(A) - 2u_c(A) + 2l_c(A).$$

(6.8)

Eqs. (6.4) and (6.5) together imply

$$\delta_c(A + B) = \delta_c(A) + \delta_c(B).$$

(6.9)

By Eqs. (6.6)–(6.8), we may also obtain

$$\delta_c(rA) = |r|\delta_c(A).$$

(6.10)

Now, let’s define four subsets of fuzzy numbers as follows:

$$\Gamma_1 := \{A | \delta_c(A) \leq 0\},$$

$$\Gamma_2 := \left\{ A | \delta_c(A) > 0, x_c(A) \geq \frac{3}{8}\delta_c(A), y_c(A) \geq \frac{3}{8}\delta_c(A) \right\},$$

$$\Gamma_3 := \left\{ A | \delta_c(A) > 0, x_c(A) < \frac{3}{8}\delta_c(A) \right\},$$

$$\Gamma_4 := \left\{ A | \delta_c(A) > 0, y_c(A) < \frac{3}{8}\delta_c(A) \right\}.$$
Let $i \geq 2$. Because $l_n, u_n, x_n, y_n$ defined as Eq. (4.3) (Eq. (4.4) or Eq. (4.5)), are linear combinations of $l_e$, $u_e$, $x_e$, $y_e$, by Eqs. (6.4) and (6.5) we obtain

\[ l_n(A + B) = l_n(A) + l_n(B), \quad u_n(A + B) = u_n(A) + u_n(B), \]
\[ x_n(A + B) = x_n(A) + x_n(B), \quad y_n(A + B) = y_n(A) + y_n(B). \]

Applying the above equalities and Eq. (6.1), we compute

\[ T_n(A + B) = (l_n(A + B), u_n(A + B); x_n(A + B), y_n(A + B)) = (l_n(A) + l_n(B), u_n(A) + u_n(B); x_n(A) + x_n(B), y_n(A) + y_n(B)) = T_n(A) + T_n(B). \]

(2) If $A \in \Gamma_1$, Eq. (6.10) implies

\[ \delta_e(rA) = |r| \delta_e(A) \leq 0. \]

Hence, $rA \in \Gamma_1$. By applying Theorem 4.4(1), we have $T_n(A) = T_d(A)$ and $T_n(rA) = T_d(rA)$. Consequently, Eq. (6.3) gives $T_n(rA) = rT_n(A)$. Hence we obtain

\[ T_n(rA) = T_n(rA) = T_n(A). \]

Let $A \in \Gamma_i$, $i \geq 2$. For $r \geq 0$, it is easy to prove $T_n(rA) = rT_n(A)$ by applying Eq. (6.6) and the fact that $l_n, u_n, x_n, y_n$ are linear combinations of $l_e, u_e, x_e, y_e$. Let $r < 0$. If $A \in \Gamma_2$, i.e. $\delta_e(A) > 0$ and $x_e(A), y_e(A) \geq \frac{3}{8} \delta_e(A)$, Eqs. (6.7) and (6.10) together imply

\[ \delta_e(rA) = -r \delta_e(A) > 0, \]
\[ x_e(rA) = -ry_e(A) \geq -r \delta_e(A) = \frac{3}{8} \delta_e(rA), \]
\[ y_e(rA) = -rx_e(A) \geq -r \delta_e(A) = \frac{3}{8} \delta_e(rA). \]

Hence, $rA \in \Gamma_2$. Substituting into Eq. (4.3) by $rA$ and $A$, we get

\[ l_n(rA) = l_e(rA) - \frac{1}{16} \delta_e(rA) = r \left[ u_e(A) + \frac{1}{16} \delta_e(A) \right] = ru_n(A), \]
\[ u_n(rA) = u_e(rA) + \frac{1}{16} \delta_e(rA) = r \left[ l_e(A) - \frac{1}{16} \delta_e(A) \right] = rl_n(A), \]
\[ x_n(rA) = x_e(rA) - \frac{3}{8} \delta_e(rA) = -r \left[ y_e(A) - \frac{3}{8} \delta_e(A) \right] = -ry_n(A), \]
\[ y_n(rA) = y_e(rA) - \frac{3}{8} \delta_e(rA) = -r \left[ x_e(A) - \frac{3}{8} \delta_e(A) \right] = -rx_n(A). \]

Hence,

\[ T_n(rA) = (l_n(rA), u_n(rA); x_n(rA), y_n(rA)) = (ru_n(A), rl_n(A); -ry_n(A), -rx_n(A)) = rT_n(A) \tag{by Eq. (6.2)}. \]

If $A \in \Gamma_3$, Eq. (6.7) implies

\[ y_e(rA) = -rx_e(A) - r \delta_e(A) = \frac{3}{8} \delta_e(rA). \]

Hence, $rA \in \Gamma_4$. Substituting into Eq. (4.5) by $rA$, we compute

\[ l_n(rA) = \frac{4}{5} l_e(rA) - \frac{1}{5} x_e(rA) = r \left[ \frac{4}{5} u_e(A) + \frac{1}{5} l_e(A) + \frac{1}{10} y_e(A) \right] = ru_n(A) \tag{by Eq. (4.4)}. \]

In the same manner, we may obtain

\[ u_n(rA) = rl_n(A), \quad x_n(rA) = -ry_n(A), \quad y_n(rA) = -rx_n(A). \]

By applying Eq. (6.2), we will obtain $T_n(rA) = rT_n(A)$. Similarly, we may prove the case $A \in \Gamma_4$. □

In the same vein, we may obtain the following proposition.
Proposition 6.2. Let A and B both belong to $C$ (or $\Gamma_1 \cup \Gamma_2$ (or $\Gamma_3$ or $\Gamma_4$). Then the nearest triangular approximation $\Delta$ satisfies:

1. $\Delta(A + B) = \Delta(A) + \Delta(B)$,
2. $\Delta(rA) = r\Delta(A)$,

for all $r \in \mathbb{R}$. Moreover, $\Delta(rA + sB) = r\Delta(A) + s\Delta(B)$ for all $r, s \in \mathbb{R}$.

Proposition 6.3. The nearest trapezoidal and nearest triangular approximations are both invariant to translations.

Proof. Let $A = [A_L(x), A_U(x)]$ be a fuzzy number, and $z$ be a real number. Then

$$A + z = [A_L(x) + z, A_U(x) + z].$$

Notice that

$$l_e(A + z) = l_e(A) + z, \quad u_e(A + z) = u_e(A) + z,$$
$$x_e(A + z) = x_e(A), \quad y_e(A + z) = y_e(A),$$
$$\delta_e(A + z) = \delta_e(A).$$

Hence, $A$ and $A + z$ belong to the same $\Gamma_i$, $1 \leq i \leq 4$. Applying Theorem 4.4 and substituting by the above equalities, we will obtain

$$T_n(A + z) = (l_n(A + z), u_n(A + z); x_n(A + z), y_n(A + z)) = (l_n(A) + z, u_n(A) + z; x_n(A), y_n(A)) = T_n(A) + z.$$

Similarly, we may apply Theorem 4.8 to prove the other one. \qed

Recall that a complete inner product space is called a Hilbert space. Let $\Omega$ be a closed convex subset of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Then there uniquely exists an element in $\Omega$ of smallest norm [14, Theorem 4.10, p. 79]. Let $\bar{a} \in H$. Hence, there uniquely exists an element in $\Omega$, denoted by $T(\bar{a})$, which minimizes the distance

$$d(\bar{a}, x) := \langle \bar{a} - x, \bar{a} - x \rangle^{1/2},$$

where $x \in \Omega$.

Fact 6.4. Let $\Omega$ be a closed convex subset of a Hilbert space $H$. Then

$$d(T(\bar{a}), T(\bar{b})) \leq d(\bar{a}, \bar{b}),$$

for all $\bar{a}, \bar{b} \in H$.

Proof. See Appendix C. \qed

Let’s define an inner product $\langle \cdot, \cdot \rangle$ for $\mathbb{R}^4$ as follows

$$\langle \bar{a}, \bar{b} \rangle := l_1l_2 + u_1u_2 + \frac{1}{12}(x_1x_2 + y_1y_2),$$

where $\bar{a} = (l_1, u_1, x_1, y_1)$ and $\bar{b} = (l_2, u_2, x_2, y_2)$. Obviously,

$$d(\bar{a}, \bar{b})^2 = \langle \bar{a} - \bar{b}, \bar{a} - \bar{b} \rangle = (l_1 - l_2)^2 + (u_1 - u_2)^2 + \frac{1}{12}[(x_1 - x_2)^2 + (y_1 - y_2)^2].$$

Every finite dimensional inner product space is Hilbert. Hence, $(\mathbb{R}^4, \langle \cdot, \cdot \rangle)$ is a Hilbert space. Let $A = (l_1, u_1; x_1, y_1)$ and $B = (l_2, u_2; x_2, y_2)$ be two extended trapezoidal fuzzy numbers. By Proposition 3.2, we obtain

$$d(A, B) = d(\bar{a}, \bar{b}).$$

This shows that Problem 4.1 (or Problem 4.7) can be represented as follows: Let $\Omega = \{(l, u, x, y)|x, y \geq 0, x + y \leq 2(u - l)\}$ (or $\Omega = \{(l, u, x, y)|x, y \geq 0, x + y = 2(u - l)\}$) and $\bar{a} = (l_e, u_e, x_e, y_e)$. Find $T(\bar{a})$. 
Obviously, $\Omega$ is a closed convex subset of $\mathbb{R}^4$. Now, applying Fact 6.4 to $T_e(A)$ and $T_e(B)$ we obtain
\[ d(T_n(A), T_n(B)) \leq d(T_e(A), T_e(B)). \] (6.11)

**Proposition 6.5.** Let $A$ and $B$ be arbitrary fuzzy numbers. Then
\[ d(T_n(A), T_n(B)) \leq d(A, B) \quad \text{and} \quad d(\Delta(A), \Delta(B)) \leq d(A, B). \]

**Proof.** Let $T_e$ denote the extended trapezoidal approximation. Proposition 3.4 gives
\[ d(T_e(A), T_e(B)) \leq d(A, B). \]
By (6.11), we get
\[ d(T_n(A), T_n(B)) \leq d(T_e(A), T_e(B)) \leq d(A, B). \]
Similarly, we may prove the other inequality by considering the closed convex subset $\{(l, u, x, y) | x, y \geq 0, x + y = 2(u - l)\}$. □

**Proposition 6.6.** The nearest trapezoidal and nearest triangular approximations are both continuous.

**Proof.** Proposition 6.5 shows that $T_n$ and $\Delta$ both satisfy the Lipschitz condition. Hence, they are continuous. □

### 7. Conclusions

The weighted triangular approximation of the fuzzy number with $\alpha$-cuts $[0, 1 - \sqrt{2}]$ is not a fuzzy number, for any weighted function. For filling the gap, we introduce the space of extended trapezoidal fuzzy numbers and extended trapezoidal approximations, so as to formulate the algorithms for computing our improved trapezoidal and triangular approximations. The two improved approximations of any fuzzy number are fuzzy numbers. Furthermore, they both satisfy translation invariance, scale invariance, identity criterion, and Lipschitz continuity.

**Acknowledgements**

The author is very grateful to the anonymous referees for their comments and suggestions, which have been very helpful in improving the presentation of this paper and formulating the algorithms in Section 5.

### Appendix A

**Proposition 3.1.** [16, Proposition 3.1]. Let $A = (l, u, x, y)$ be an extended trapezoidal fuzzy number. Then

1. $A$ is trapezoidal iff $x, y \geq 0$ and $x + y \leq 2(u - l)$.
2. $A$ is triangular iff $x, y \geq 0$ and $x + y = 2(u - l)$.

**Proof.** Obviously (3.1) implies
\[ A = \left[ l + x\left(\frac{u - 1}{2}\right), u - y\left(\frac{x - 1}{2}\right) \right] = \left[ l - \frac{1}{2}x + x\alpha, u + \frac{1}{2}y - y\alpha \right]. \]
By applying the constraint (2.1), $A$ is trapezoidal iff
\[ l - \frac{1}{2}x \leq l + \frac{1}{2}x \leq u - \frac{1}{2}y \leq u + \frac{1}{2}y. \]
It is easily verified that the above inequalities are equivalent to \( x, y \geq 0 \) and \( x + y \leq 2(u - l) \). In the same vein, we may get that \( A \) is a triangular fuzzy number iff \( x, y \geq 0 \) and \( x + y = 2(u - l) \). □

Appendix B

**Proposition 3.2.** [16, Proposition 3.3] Let \( A = (l_1, u_1; x_1, y_1) \) and \( B = (l_2, u_2; x_2, y_2) \) be two extended trapezoidal fuzzy numbers. Then

\[
d(A, B)^2 = (l_1 - l_2)^2 + (u_1 - u_2)^2 + \frac{1}{12}[(x_1 - x_2)^2 + (y_1 - y_2)^2].
\]

**Proof.** Obviously, (3.1) implies

\[
A = \left[ l_1 + x_1 \left( \alpha - \frac{1}{2} \right), u_1 - y_1 \left( \alpha - \frac{1}{2} \right) \right], B = \left[ l_2 + x_2 \left( \alpha - \frac{1}{2} \right), u_2 - y_2 \left( \alpha - \frac{1}{2} \right) \right].
\]

From Eq. (1.1), we compute

\[
d(A, B)^2 = \int_0^1 \left| l_1 + x_1 \left( \alpha - \frac{1}{2} \right) - \left( l_2 + x_2 \left( \alpha - \frac{1}{2} \right) \right) \right|^2 \, d\alpha
\]

\[
+ \int_0^1 \left| u_1 - y_1 \left( \alpha - \frac{1}{2} \right) - \left( u_2 - y_2 \left( \alpha - \frac{1}{2} \right) \right) \right|^2 \, d\alpha
\]

\[
= \int_0^1 (l_1 - l_2)^2 \, d\alpha + \int_0^1 (x_1 - x_2)^2 \left( \alpha - \frac{1}{2} \right)^2 \, d\alpha + \int_0^1 (y_1 - y_2)^2 \left( \alpha - \frac{1}{2} \right)^2 \, d\alpha
\]

\[
= (l_1 - l_2)^2 + (u_1 - u_2)^2 + \frac{1}{12}[(x_1 - x_2)^2 + (y_1 - y_2)^2].
\]

This completes the proof. □

Appendix C

Let \( (H, \langle \cdot, \cdot \rangle) \) be a Hilbert space, \( \Omega \) be a closed convex subset of \( H \), \( \bar{a} \in H \), and let \( T(\bar{a}) \) denote the element in \( \Omega \) which minimizes the distance

\[
d(\bar{a}, \bar{x}) := \langle \bar{a} - \bar{x}, \bar{a} - \bar{x} \rangle^{\frac{1}{2}},
\]

where \( \bar{x} \in \Omega \). Then \( d(T(\bar{a}), T(\bar{b})) \leq d(\bar{a}, \bar{b}), \) for all \( \bar{a}, \bar{b} \in H \).

**Proof.** Let \( t \in (0, 1) \). We get \((1 - t)T(\bar{a}) + tT(\bar{b}) \in \Omega \), since \( \Omega \) is convex. Obviously,

\[
d(\bar{a}, (1 - t)T(\bar{a}) + tT(\bar{b}))^2 = \langle \bar{a} - (1 - t)T(\bar{a}) + tT(\bar{b}), \bar{a} - T(\bar{a}) + t[T(\bar{a}) - T(\bar{b})] \rangle
\]

\[
= d(\bar{a}, T(\bar{a}))^2 + 2t\langle \bar{a} - T(\bar{a}), T(\bar{a}) - T(\bar{b}) \rangle + t^2 d(T(\bar{a}), T(\bar{b}))^2.
\]

The definition of \( T(\bar{a}) \) implies \( d(\bar{a}, (1 - t)T(\bar{a}) + tT(\bar{b})) \geq d(\bar{a}, T(\bar{a})) \) for all \( t \in (0, 1) \). That is

\[
2t\langle \bar{a} - T(\bar{a}), T(\bar{a}) - T(\bar{b}) \rangle + t^2 d(T(\bar{a}), T(\bar{b}))^2 \geq 0,
\]

or equivalently,

\[
\langle \bar{a} - T(\bar{a}), T(\bar{a}) - T(\bar{b}) \rangle \geq -\frac{t}{2} d(T(\bar{a}), T(\bar{b}))^2.
\]

Let \( t \to 0^+ \), we obtain \( \langle \bar{a} - T(\bar{a}), T(\bar{a}) - T(\bar{b}) \rangle \geq 0 \). Similarly, we may also obtain \( \langle \bar{b} - T(\bar{b}), T(\bar{b}) - T(\bar{a}) \rangle \geq 0 \). Let

\[
\bar{d} := \bar{a} - T(\bar{a}) - (\bar{b} - T(\bar{b})),
\]
we conclude
\[ \langle \tilde{d}, T(\tilde{a}) - T(\tilde{b}) \rangle \geq 0. \]

Now, let’s compute
\[
\langle \tilde{a} - \tilde{b}, \tilde{a} - \tilde{b} \rangle = \langle \tilde{d} + T(\tilde{a}) - T(\tilde{b}), \tilde{d} + T(\tilde{a}) - T(\tilde{b}) \rangle \\
= \langle \tilde{d}, \tilde{d} \rangle + 2\langle \tilde{d}, T(\tilde{a}) - T(\tilde{b}) \rangle + \langle T(\tilde{a}) - T(\tilde{b}), T(\tilde{a}) - T(\tilde{b}) \rangle \geq \langle T(\tilde{a}) - T(\tilde{b}), T(\tilde{a}) - T(\tilde{b}) \rangle.
\]

This completes the proof. \(\square\)

References