Discontinuity of the trapezoidal fuzzy number-valued operators preserving core

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\textbf{ABSTRACT}

We prove that any trapezoidal fuzzy number-valued operator preserving core is discontinuous with respect to any weighted metric on the space of fuzzy numbers. As an application, we obtain the discontinuity of the weighted trapezoidal approximation operator preserving core.

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1. Introduction

The continuity is considered between the essential properties of an approximation operator. In applications, where it is sometimes indicated as robustness, the criterion of continuity is of extreme importance too. In the present paper we prove that any trapezoidal fuzzy number-valued operator preserving core is discontinuous with respect to any weighted metric and each fuzzy number with 1-cut set as a proper interval is a point of discontinuity. As an immediate consequence, the result of continuity of the trapezoidal approximation operator preserving core in [1] is not valid. Nevertheless, this operator has a good property proved in the final part of the paper. These results are important because continuity is between the criteria that a trapezoidal (triangular, parametric) approximation operator should possesses (see [2]). In the past years many researchers have focused on finding such approximations of fuzzy numbers, with respect to average Euclidean or weighted metrics, with or without additional conditions (see [1–11]). The aim is the simplification of calculus for easy interpretation and implementation, under conditions of preservation of information, in applications related with multiple areas where the fuzzy numbers are used to represent uncertain and incomplete information: decision making, linguistic controllers, biotechnological systems, expert systems, data mining, pattern recognition, etc.

2. Preliminaries

We consider the following well-known description of a fuzzy number $A$:

$$A(x) = \begin{cases} 0, & \text{if } x \leq a_1 \\ l_A(x), & \text{if } a_1 \leq x \leq a_2 \\ 1, & \text{if } a_2 \leq x \leq a_3 \\ r_A(x), & \text{if } a_3 \leq x \leq a_4 \\ 0, & \text{if } a_4 \leq x, \end{cases}$$

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where \(a_1, a_2, a_3, a_4 \in \mathbb{R}\), \(l_A : [a_1, a_2] \rightarrow [0, 1]\) is a nondecreasing upper semicontinuous function, \(l_A(a_1) = 0, l_A(a_2) = 1\), called the left side of \(A\) and \(r_A : [a_3, a_4] \rightarrow [0, 1]\) is a nonincreasing upper semicontinuous function, \(r_A(a_3) = 1, r_A(a_4) = 0\), called the right side of \(A\). The \(\alpha\)-cut, \(\alpha \in (0, 1)\), of a fuzzy number \(A\) is a crisp set defined as

\[ A_\alpha = \{ x \in \mathbb{R} : A(x) \geq \alpha \}. \]

The support or \(0\)-cut \(A_0\) of \(A\) is defined as

\[ A_0 = \text{cl}\{ x \in \mathbb{R} : A(x) > 0 \}, \]

where \(\text{cl}\) is the closure operator. Every \(\alpha\)-cut, \(\alpha \in [0, 1]\), of \(A\) is a closed interval

\[ A_\alpha = [A_L(\alpha), A_U(\alpha)]. \]

where

\[ A_L(\alpha) = \inf\{ x \in \mathbb{R} : A(x) \geq \alpha \}, \]

\[ A_U(\alpha) = \sup\{ x \in \mathbb{R} : A(x) \geq \alpha \}, \]

for any \(\alpha \in (0, 1]\). The core of \(A\) is defined as

\[ \text{core}(A) = A_1 = [A_L(1), A_U(1)]. \]

If the sides of the fuzzy number \(A\) are strictly monotone then one can see easily that \(A_L\) and \(A_U\) are inverse functions of \(l_A\) and \(r_A\), respectively. We denote by \(F(\mathbb{R})\) the set of all fuzzy numbers.

The average Euclidean metric \(d_{1,1}\) on \(F(\mathbb{R})\) is defined by [12]

\[ d_{1,1}(A, B) = \int_0^1 (A_L(\alpha) - B_L(\alpha))^2 d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 d\alpha. \tag{2} \]

A weighted metric \(d_{f,g}\) on \(F(\mathbb{R})\), which generalizes the above metric, is defined by (see [10])

\[ d_{f,g}(A, B) = \int_0^1 f(\alpha)(A_L(\alpha) - B_L(\alpha))^2 d\alpha + \int_0^1 g(\alpha)(A_U(\alpha) - B_U(\alpha))^2 d\alpha, \tag{3} \]

where \(f, g : [0, 1] \rightarrow \mathbb{R}\) are called weighted functions, that is they are integrable, nonnegative, nondecreasing and satisfy \(\int_0^1 f(\alpha) d\alpha > 0\) and \(\int_0^1 g(\alpha) d\alpha > 0\). The property of monotonicity of functions \(f\) and \(g\) means that the higher the cut level is, the more important its weight is in determining the distance of fuzzy numbers \(A\) and \(B\). Often \(f(\alpha) = g(\alpha) = 1\), for every \(\alpha \in [0, 1]\), and the average Euclidean metric is obtained or \(f(\alpha) = g(\alpha) = \alpha\), for every \(\alpha \in [0, 1]\). Sometimes the additional conditions \(f(0) = g(0) = 0\) and \(\int_0^1 f(\alpha) d\alpha = \int_0^1 g(\alpha) d\alpha = \frac{1}{2}\) are imposed to weighted functions (see [11]).

Fuzzy numbers with simple membership functions are preferred in practice. The most often used fuzzy numbers are so-called trapezoidal fuzzy numbers. A trapezoidal fuzzy number \(T, T_\alpha = [T_L(\alpha), T_U(\alpha)], \alpha \in [0, 1]\), is given by

\[ T_L(\alpha) = t_1 + (t_2 - t_1)\alpha \]

and

\[ T_U(\alpha) = t_4 - (t_4 - t_3)\alpha, \]

with \(t_1 \leq t_2 \leq t_3 \leq t_4\). We denote

\[ T = (t_1, t_2, t_3, t_4) \]

a trapezoidal fuzzy number and by \(F^T(\mathbb{R})\) the set of all trapezoidal fuzzy numbers.

A trapezoidal fuzzy number-valued operator \(T : F(\mathbb{R}) \rightarrow F^T(\mathbb{R})\) is called (see [2]) continuous with respect to metric \(d : F(\mathbb{R}) \times F(\mathbb{R}) \rightarrow [0, +\infty)\) if, for any \(A, B \in F(\mathbb{R})\), we have

\[ \forall \epsilon > 0, \exists \delta > 0 : d(A, B) < \delta \implies d(T(A), T(B)) < \epsilon. \]

3. Auxiliary results

To prove the main results of the paper, first we need some auxiliary results as follows.

**Lemma 1.** Let \(f : [0, 1] \rightarrow \mathbb{R}\) be a weighted function. Then

\[ \left( \int_0^1 f(\alpha)\alpha(1-\alpha) d\alpha \right)^2 < \int_0^1 f(\alpha)\alpha^2 d\alpha \int_0^1 f(\alpha)(1-\alpha)^2 d\alpha. \]
Proof. According to Theorem 3.2 in [1] we have (the Schwarz inequality is used)

\[
\left( \int_0^1 f(\alpha)\alpha(1-\alpha)d\alpha \right)^2 \leq \int_0^1 f(\alpha)^2d\alpha \int_0^1 (1-\alpha)^2d\alpha.
\]

Let us suppose that in the above inequality we have equality. Then there exists a constant \( k > 0 \) such that \( \sqrt{f(\alpha)\alpha} = k\sqrt{f(\alpha)/(1-\alpha)} \), almost everywhere \( \alpha \in [0, 1] \) (we cannot have \( k = 0 \) because this would imply \( \sqrt{f(\alpha)\alpha} = 0 \), almost everywhere \( \alpha \in [0, 1] \), a contradiction with \( \int_0^1 f(\alpha)d\alpha > 0 \)). Since \( f \) is nondecreasing and \( \int_0^1 f(\alpha)d\alpha > 0 \), it follows that there exists \( c \in (0, 1) \) such that \( f(c) > 0 \) which implies \( \alpha = k(1-\alpha) \), almost everywhere on the interval \( [c, 1] \). This immediately implies \( \alpha(1+k) = k \), almost everywhere on the interval \( [c, 1] \), a contradiction. \( \Box \)

Lemma 2. If \( (T_n)_{n \in \mathbb{N}} \), \( T_n = (t_1(n), t_2(n), t_3(n), t_4(n)) \) is a sequence of trapezoidal fuzzy numbers such that \( \lim_{n \to \infty} t_i(n) = t_i < \infty \), \( \forall i \in \{1, 2, 3, 4\} \), then \( \lim_{n \to \infty} T_n = T \) with respect to any weighted metric \( d_{f,g} \), where \( T \) is the trapezoidal fuzzy number \( T = (t_1, t_2, t_3, t_4) \).

Proof. By direct calculation we get

\[
d^2_{f,g}(T, T_n) = \int_0^1 f(\alpha)\left((T_n)_L(\alpha) - T_L(\alpha)\right)^2d\alpha + \int_0^1 g(\alpha)\left((T_n)_U(\alpha) - T_U(\alpha)\right)^2d\alpha
= \int_0^1 f(\alpha)(1-\alpha)2d\alpha \int_0^1 (2(t_1(n) - t_1)(t_2(n) - t_2)
\quad \times \int_0^1 f(\alpha)(1-\alpha)2d\alpha + (t_3(n) - t_3)^2 \int_0^1 g(\alpha)(1-\alpha)2d\alpha + (t_4(n) - t_4)^2 \int_0^1 g(\alpha)(1-\alpha)2d\alpha
\quad + 2(t_4(n) - t_4)(t_3(n) - t_3) \int_0^1 g(\alpha)(1-\alpha)2d\alpha.
\]

Since by the hypothesis \( \lim_{n \to \infty} t_i(n) = t_i, i \in \{1, 2, 3, 4\} \) it is immediate that \( \lim_{n \to \infty} d^2_{f,g}(T, T_n) = 0 \). This implies \( \lim_{n \to \infty} T_n = T \) (with respect to \( d_{f,g} \)). \( \Box \)

Lemma 3. If \( (T_n)_{n \in \mathbb{N}} \), \( T_n = (t_1(n), t_2(n), t_3(n), t_4(n)) \) is a sequence of trapezoidal fuzzy numbers convergent with respect to a weighted metric \( d_{f,g} \), then its limit is a trapezoidal fuzzy number \( T = (t_1, t_2, t_3, t_4) \), where \( t_i = \lim_{n \to \infty} t_i(n), i \in \{1, 2, 3, 4\} \).

Proof. For simplicity, let us denote

\[
\begin{align*}
u &= \frac{\int_0^1 f(\alpha)\alpha(1-\alpha)d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2d\alpha} - \left( \int_0^1 f(\alpha)\alpha(1-\alpha)d\alpha \right)^2, \\
v &= \frac{\int_0^1 f(\alpha)\alpha(1-\alpha)d\alpha}{\int_0^1 f(\alpha)(1-\alpha)^2d\alpha} - \left( \int_0^1 f(\alpha)\alpha(1-\alpha)d\alpha \right)^2, \\
c_{n,m} &= \frac{(t_2(n) - t_2(m))}{\int_0^1 f(\alpha)(1-\alpha)^2d\alpha}, \\
d_{n,m} &= \frac{(t_1(n) - t_1(m))}{\int_0^1 f(\alpha)(1-\alpha)^2d\alpha},
\end{align*}
\]

and for all \( m, n \in \mathbb{N} \). Note that \( u, v, c_{n,m} \) and \( d_{n,m} \) are correctly defined. Indeed, as in the proof of Lemma 1, let \( c \in (0, 1) \) be such that \( f(c) > 0 \). Then

\[
\int_0^1 f(\alpha)(1-\alpha)^2d\alpha \geq f(c) \int_c^1 (1-\alpha)^2d\alpha = f(c) \frac{1-c^3}{3} > 0
\]

and

\[
\int_0^1 f(\alpha)(1-\alpha)^2d\alpha \geq f(c) \int_c^1 (1-\alpha)^2d\alpha = f(c) \frac{(1-c)^3}{3} > 0.
\]

Lemma 1 implies \( u > 0 \) and \( v > 0 \).
Now, let us fix $\varepsilon > 0$. Since the sequence $(T_n)_{n \in \mathbb{N}}$ is convergent, it follows that there exists $n_0 \in \mathbb{N}$ such that

$$d_{f,g}^2(T_m, T_n) \leq \min \left\{ \frac{u\varepsilon^2}{2}, \frac{v\varepsilon^2}{2} \right\}, \quad \forall n \geq n_0, \forall m \geq n_0.$$ 

This implies

$$\int_0^1 f(\alpha) ((T_n)_L(\alpha) - (T_m)_L(\alpha))^2 \, d\alpha \leq \min \left\{ \frac{u\varepsilon^2}{2}, \frac{v\varepsilon^2}{2} \right\}$$

and

$$\int_0^1 g(\alpha) ((T_n)_U(\alpha) - (T_m)_U(\alpha))^2 \, d\alpha \leq \min \left\{ \frac{u\varepsilon^2}{2}, \frac{v\varepsilon^2}{2} \right\},$$

for all $n \geq n_0$ and $m \geq n_0$. By direct calculation we get

$$\int_0^1 f(\alpha) ((T_n)_L(\alpha) - (T_m)_L(\alpha))^2 \, d\alpha$$

$$= (t_1(n) - t_1(m))^2 \int_0^1 f(\alpha)(1 - \alpha)^2 \, d\alpha + (t_2(n) - t_2(m))^2 \int_0^1 f(\alpha)\alpha^2 \, d\alpha + 2(t_1(n) - t_1(m))(t_2(n) - t_2(m)) \int_0^1 f(\alpha)(1 - \alpha)\alpha \, d\alpha$$

$$\leq \left[ (t_1(n) - t_1(m)) + c_{n,m} \right] \frac{1}{2} \int_0^1 f(\alpha)(1 - \alpha)^2 \, d\alpha + \frac{u(t_2(n) - t_2(m))^2}{2}$$

$$+ \left[ (t_2(n) - t_2(m)) + d_{n,m} \right] \frac{1}{2} \int_0^1 f(\alpha)\alpha^2 \, d\alpha + \frac{v(t_1(n) - t_1(m))^2}{2},$$

from the equality $ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a}$, first for $x = t_1(n) - t_1(m)$ and then for $x = t_2(n) - t_2(m)$. Relation (4) implies

$$|t_1(n) - t_1(m)| < \varepsilon$$

and

$$|t_2(n) - t_2(m)| < \varepsilon$$

for all $n, m \in \mathbb{N}$, $n \geq n_0$, $m \geq n_0$. Consequently, the sequences $(t_1(n))_{n \in \mathbb{N}}$ and $(t_2(n))_{n \in \mathbb{N}}$ are convergent and there exist real numbers $t_1$ and $t_2$ such that $\lim_{n \to \infty} t_1(n) = t_1$ and $\lim_{n \to \infty} t_2(n) = t_2$.

In the same way one can prove that there exist real numbers $t_3$ and $t_4$ such that $\lim_{n \to \infty} t_3(n) = t_3$ and $\lim_{n \to \infty} t_4(n) = t_4$. Clearly, $T = (t_1, t_2, t_3, t_4)$ is a trapezoidal fuzzy number and by Lemma 2, we immediately obtain that $\lim_{n \to \infty} T_n = T$, with respect to $d_{f,g}$. \(\square\)

4. Discontinuity of the trapezoidal fuzzy number-valued operators preserving core

The following result proves that the behavior of the trapezoidal fuzzy number-valued operators with the additional condition to preserve the core is unnatural.

**Theorem 4.** Let $T : F(\mathbb{R}) \to F^T(\mathbb{R})$ be a trapezoidal fuzzy number-valued operator preserving core, that is $\text{core}(A) = \text{core} \, T(A)$, for every $A \in F(\mathbb{R})$. If $A \in F(\mathbb{R})$, $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, $\alpha \in [0, 1]$, satisfies $A_L(1) < A_U(1)$, then $T$ is discontinuous in $A$ with respect to any weighted metric $d_{f,g}$.

**Proof.** Let us consider $A \in F(\mathbb{R})$ such that $A_L(1) < A_U(1)$ and the sequence of fuzzy numbers $(A_n)_{n \in \mathbb{N}}$ given by

$$(A_n)_L(\alpha) = A_L(\alpha) + \alpha^3 (A_U(1) - A_L(1)),$$

$$(A_n)_U(\alpha) = A_U(\alpha), \quad \alpha \in [0, 1].$$

It is easy to check that the function $(A_n)_L$ is nondecreasing and

$$(A_n)_L(1) = A_L(1) + (A_U(1) - A_L(1)) = A_U(1) = (A_n)_U(1).$$

Therefore, we have $A_L \leq (A_n)_L \leq (A_n)_U \leq A_U$, for all $n \in \mathbb{N}$ and $\alpha \in [0, 1]$. Consequently, $A_n \to A$, $n \to \infty$. However, $T(A_n) \to T(A)$ for $n \to \infty$. Hence, $T$ is discontinuous in $A$. \(\square\)
Theorem 4. Indeed, if follows that there exists a nearest \((A_n)_{\leq 1} = A_U(1) > A_L(1) = t_2\), and therefore (see Lemma 3) we cannot have \(\lim_{n \to \infty} T(A_n) = T(A)\) with respect to weighted metric \(d_{f, g}\). By Heine’s criterion it follows that \(T\) is discontinuous in \(A\) and the proof is complete. 

5. Case of weighted trapezoidal approximation operator preserving core

Considering that the core of a fuzzy number is an important parameter in real problems, in [1] the following result was proposed.

Theorem 5 ([1], (3.15)). Let \(A, A_\# = [A_L(\alpha), A_U(\alpha)], \alpha \in [0, 1]\), be a fuzzy number and

\[
T_{c, d_{f, g}}(A) = (t_1(A), t_2(A), t_3(A), t_4(A)) = (t_1, t_2, t_3, t_4)
\]

be the nearest (with respect to the weighted metric \(d_{f, g}\)) trapezoidal fuzzy number to fuzzy number \(A\) which preserves the core. Then

\[
t_1 = \frac{-\int_0^1 (\alpha - 1)A_U(\alpha)f(\alpha)d\alpha + A_L(1)\int_0^1 \alpha(\alpha - 1)f(\alpha)d\alpha}{\int_0^1 (\alpha - 1)^2 f(\alpha)d\alpha},
\]

\[
t_2 = A_L(1),
\]

\[
t_3 = A_U(1),
\]

\[
t_4 = \frac{-\int_0^1 (\alpha - 1)A_U(\alpha)f(\alpha)d\alpha + A_U(1)\int_0^1 \alpha(\alpha - 1)f(\alpha)d\alpha}{\int_0^1 (\alpha - 1)^2 f(\alpha)d\alpha}.
\]

In [1], Theorem 3.6 the authors claimed that \(T_{c, d_{1, 1}}\) is continuous with respect to metric \(d_{1, 1}\). This conclusion is wrong. According to Theorem 4 the operator \(T_{c, d_{1, 1}} : F(\mathbb{R}) \to F^T(\mathbb{R})\) is discontinuous in any fuzzy number \(A\) satisfying \(A_L(1) < A_U(1)\), with respect to any weighted metric \(d_{g, h}\). In addition,

Theorem 6. \(T_{c, d_{f, g}} : F(\mathbb{R}) \to F^T(\mathbb{R})\) is discontinuous with respect to any weighted metric \(d_{g, h}\).

Remark 7. The error is in the last inequality of the proof of Theorem 3.6, [1], that is

\[
3 \int_0^1 (\alpha - 1)^2 (A_L(\alpha) - B_L(\alpha))^2 d\alpha + \frac{1}{4} (A_L(1) - B_L(1))^2
\]

\[
+ 3 \int_0^1 (\alpha - 1)^2 (A_U(\alpha) - B_U(\alpha))^2 d\alpha + \frac{1}{4} (A_U(1) - B_U(1))^2
\]

\[
\leq 3d_{1, 1}^2(A, B).
\]

Indeed, if \(A \in F(\mathbb{R})\) satisfies \(A_U(1) > A_L(1)\) and \(B \in F(\mathbb{R})\) is given by

\[
B_L(\alpha) = A_L(\alpha) + \alpha^6 (A_U(1) - A_L(1)),
\]

\[
B_U(\alpha) = A_U(\alpha), \quad \alpha \in [0, 1]
\]
then we obtain the left side of (6)
\[
3 \int_0^1 \alpha^2 (\alpha - 1)^2 (A_U(1) - A_L(1))^2 \, d\alpha + \frac{1}{4} (A_U(1) - A_L(1))^2
\]
\[
= \frac{459}{1820} (A_U(1) - A_L(1))^2 > \frac{1}{4} (A_U(1) - A_L(1))^2
\]
and the right side of (6)
\[
3 \int_0^1 (A_L(\alpha) - B_L(\alpha))^2 \, d\alpha + 3 \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 \, d\alpha
\]
\[
= \frac{3}{13} (A_U(1) - A_L(1))^2 < \frac{1}{4} (A_U(1) - A_L(1))^2,
\]
therefore a contradiction with (6).

From the computational point of view, the discontinuity of \( T_{c,dj} \) is an important disadvantage. If the calculus in (5) is inapplicable for a fuzzy number \( A \) (we cannot compute exactly the integrals), then we cannot use the approximation theory methods to find \( T_{c,dj}(A) \) within a reasonable error. Indeed, if the fuzzy number \( B \) is close to fuzzy number \( A \), then one would expect that \( T_{c,dj}(B) \) is close to \( T_{c,dj}(A) \), but in the absence of continuity this is not necessarily true.

To overcome the handicap of the discontinuity of the operator \( T_{c,dj} \) we present the following result.

**Theorem 8.** If \( A, A_a = [A_L(\alpha), A_U(\alpha)], \alpha \in [0, 1] \), is a fuzzy number and \((A_n)_{n \in \mathbb{N}}\) is a sequence of fuzzy numbers such that \((A_n)_{n \in \mathbb{N}}\) and \((A_n)_{n \in \mathbb{N}}\) are uniform convergent sequences to \( A_L \) and \( A_U \), respectively, then
\[
\lim_{n \to \infty} T_{c,dj}(A_n) = T_{c,dj}(A),
\]
with respect to any weighted metric \( d_{g,h} \).

**Proof.** Let us denote
\[
T_{c,dj}(A) = (t_1, t_2, t_3, t_4)
\]
and
\[
T_{c,dj}(A_n) = (t_1(n), t_2(n), t_3(n), t_4(n)), \quad n \in \mathbb{N}.
\]
The uniform convergence of \((A_n)_{n \in \mathbb{N}}\) to \( A_L \) implies the uniform convergence of \((P_n)_{n \in \mathbb{N}}\) to \( P_L \), where
\[
(P_n)_{\alpha} = (\alpha - 1) (A_n)_{\alpha} f(\alpha), \quad \alpha \in [0, 1], \quad n \in \mathbb{N}
\]
and
\[
P_L(\alpha) = (\alpha - 1) A_L(\alpha) f(\alpha), \quad \alpha \in [0, 1].
\]
We have
\[
\lim_{n \to \infty} \int_0^1 (\alpha - 1) (A_n)_{\alpha} f(\alpha) \, d\alpha = \int_0^1 (\alpha - 1) A_L(\alpha) f(\alpha) \, d\alpha
\]
and
\[
\lim_{n \to \infty} (A_n)_{\alpha}(1) = A_L(1).
\]
We obtain (see (5))
\[
\lim_{n \to \infty} t_1(n) = \lim_{n \to \infty} \frac{- \int_0^1 (\alpha - 1) (A_n)_{\alpha} f(\alpha) \, d\alpha + (A_n)_{\alpha}(1) \int_0^1 \alpha(\alpha - 1) f(\alpha) \, d\alpha}{\int_0^1 (\alpha - 1)^2 f(\alpha) \, d\alpha}
\]
\[
= \lim_{n \to \infty} \frac{- \int_0^1 (\alpha - 1) (A_n)_{\alpha} f(\alpha) \, d\alpha + \int_0^1 \alpha(\alpha - 1) f(\alpha) \, d\alpha}{\int_0^1 (\alpha - 1)^2 f(\alpha) \, d\alpha}
\]
\[
= \frac{- \int_0^1 (\alpha - 1) A_L(\alpha) f(\alpha) \, d\alpha + A_L(1) \int_0^1 \alpha(\alpha - 1) f(\alpha) \, d\alpha}{\int_0^1 (\alpha - 1)^2 f(\alpha) \, d\alpha}
\]
\[
= t_1 < \infty.
\]
and
\[
\lim_{n \to \infty} t_2(n) = \lim_{n \to \infty} (A_n)_L(1) = A_L(1) = t_2 < \infty.
\]

In a similar way
\[
\lim_{n \to \infty} t_3(n) = t_3 < \infty,
\]
\[
\lim_{n \to \infty} t_4(n) = t_4 < \infty
\]
and according to Lemma 2 we get
\[
\lim_{n \to \infty} T_{c,d_{f,f}}(A_n) = T_{c,d_{f,f}}(A),
\]
with respect to weighted metric \( d_{g,h} \).

We exemplify below how the trapezoidal approximation preserving core of a fuzzy number can be estimated if the integrals in (5) cannot be calculated and Theorem 8 is applicable.

**Example 9.** Let us consider the fuzzy number \( A \), given by

\[
A_L(\alpha) = \frac{1}{2} e^{\alpha^2},
\]
\[
A_U(\alpha) = 5 - \alpha, \quad \alpha \in [0, 1].
\]
The sequence of fuzzy numbers \( (A_n)_{n \in \mathbb{N}} \) given by

\[
(A_n)_L(\alpha) = \frac{1}{2} \left( 1 + \alpha^2 + \frac{\alpha^4}{2!} + \cdots + \frac{\alpha^{2n}}{n!} \right),
\]
\[
(A_n)_U(\alpha) = 5 - \alpha, \quad \alpha \in [0, 1],
\]
satisfies the hypothesis in Theorem 8. We have

\[
e^{\alpha^2} = 1 + \alpha^2 + \frac{\alpha^4}{2!} + \frac{\alpha^6}{3!} + \cdots + \frac{\alpha^{2n}}{n!} + \int_0^{\alpha^2} \frac{(\alpha^2 - t)^n}{n!} e^t dt
\]

and

\[
0 \leq A_L(\alpha) - (A_n)_L(\alpha)
\]
\[
= \frac{1}{2} \int_0^{\alpha^2} \frac{(\alpha^2 - t)^n}{n!} e^t dt
\]
\[
= \frac{e}{2} \frac{\alpha^{2n+2}}{(n+1)!} \leq \frac{e}{2(n+1)!},
\]
for every \( \alpha \in [0, 1] \).

Now, let

\[
T_{c,d_{1,1}}(A) = (t_1, t_2, t_3, t_4)
\]
and

\[
T_{c,d_{1,1}}(A_n) = (t_1(n), t_2(n), t_3(n), t_4(n)), \quad n \in \mathbb{N},
\]
the trapezoidal approximation preserving core of \( A \) and \( A_n \), respectively. We obtain

\[
d^2_{1,1}(T_{c,d_{1,1}}(A), T_{c,d_{1,1}}(A_n)) = \int_0^1 \left( (T_{c,d_{1,1}}(A))_L(\alpha) - (T_{c,d_{1,1}}(A_n))_L(\alpha) \right)^2 d\alpha
\]
\[
+ \int_0^1 \left( (T_{c,d_{1,1}}(A))_U(\alpha) - (T_{c,d_{1,1}}(A_n))_U(\alpha) \right)^2 d\alpha
\]
\[
= \frac{1}{3} (t_1(n) - t_1)^2 + \frac{1}{3} (t_1(n) - t_1)(t_2(n) - t_2) + \frac{1}{3} (t_2(n) - t_2)^2
\]
\[
\leq \frac{1}{3} (t_1(n) - t_1)^2 + \frac{1}{3} |t_1(n) - t_1| \times |t_2(n) - t_2| + \frac{1}{3} (t_2(n) - t_2)^2.
\]
Because (see (5) and (7))

\[ |t_2(n) - t_2| = |A_2(1) - (A_n)_2(1)| \leq \frac{e}{2(n + 1)!} \]

and

\[ |t_1(n) - t_1| = \left| \frac{1}{2} (A_1(1) - (A_n)_1(1)) + 3 \int_0^1 (1 - \alpha) (A_1(\alpha) - (A_n)_1(\alpha)) \, d\alpha \right| \]

\[ \leq \frac{1}{2} |A_1(1) - (A_n)_1(1)| + 3 \int_0^1 (1 - \alpha) (A_1(\alpha) - (A_n)_1(\alpha)) \, d\alpha \]

\[ \leq \frac{e}{4(n + 1)!} + \frac{3e}{2(n + 1)!} = \frac{7e}{4(n + 1)!} \]

we get

\[ d^2_{1,1}(T_{c,d_1,1}(A), T_{c,d_1,1}(A_n)) \leq \frac{49e^2}{48 ((n + 1))!^2} + \frac{e^2}{12 ((n + 1))!^2} + \frac{7e^2}{24 ((n + 1))!^2} \]

\[ = \frac{67e^2}{48 ((n + 1))!^2} \leq 10^{-4}, \]

for all \( n \geq 5 \). Therefore, \( T_{c,d_1,1}(A_n) \) approximates \( T_{c,d_1,1}(A) \) with an error less than \( 10^{-2} \) with respect to the metric \( d_{1,1} \). Applying the method in Theorem 5 we obtain

\[ T_{c,d_1,1}(A_n) = \left( \frac{50 \, 137 \, 163}{221 \, 760}, 4, 5 \right). \]

6. Conclusions

In the present contribution we prove (Theorem 4) the discontinuity of trapezoidal fuzzy number-valued operators preserving core with respect to any weighted metric. This implies some difficulties from the computational point of view and problems of robustness in applications. We notice that the continuity of the weighted trapezoidal approximation operator preserving core, proved in a recent article, is not valid (see Theorem 6). Nevertheless, the weighted trapezoidal approximation operator preserving core has a relative good behavior, pointed out in Theorem 8 and applied in Example 9.

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